PRACTICAL COMPUTATIONAL METHODS IN THE SOLUTION OF EQUATIONS*

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1. Introduction. The basically simple techniques of this paper greatly shorten the labor in computing solutions of ordinary linear differential equations with constant coefficients and algebraic equations arising in applied problems. The reader is especially referred to Section 3 on complex roots.

We shall be concerned with some practical methods of approximating the roots of the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \tag{1}$$

in which the coefficients $a_1, a_2, \cdots, a_n$ are real. This equation is associated with the linear differential equation

$$\frac{d^ny}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_ny = 0. \tag{2}$$

We often wish to solve (2) with the initial conditions

$$t = 0,\ y = A_1,\ \frac{dy}{dt} = A_2, \cdots,\ \frac{d^{n-1}y}{dt^{n-1}} = A_n \tag{3}$$

in which $A_1, A_2, \cdots, A_n$ are real.

From probability considerations we can say that if (2) is a differential equation which represents a physical system it is very unusual for two roots of the corresponding equation (1) to be equal or almost equal. It is therefore assumed in this paper that the roots $\alpha_1, \alpha_2, \cdots, \alpha_n$ of (1) are distinct. Then there are numbers $B_1, B_2, \cdots, B_n$ such that the solution of(2) and (3) is

$$y = B_1e^{\alpha_1t} + B_2e^{\alpha_2t} + \cdots + B_ne^{\alpha_nt}. \tag{4}$$

If i is the imaginary unit, there are real numbers u, v such that $\alpha_1 = u+iv$. Then the term $B_1e^{\alpha_1t}$ in (4) is $B_1e^{ut}(\cos vt + i \sin vt)$. If $B_1 \neq 0$, and u is negative, the term B_1e^{ut} approaches zero as t increases beyond all bounds; otherwise B_1e^{ut} does not approach zero. For this reason, if t is time, (2) is said to be the differential equation of a stable physical system if and only if the real parts of the roots of (1) are all negative. In the design of any piece of equipment the question of stability is an important one.

If $n \geqq 3$ the equation (1) should be solved by trial in preference to any standard method with prescribed steps. Some information about the nature of the roots should be obtained before carrying out the actual solution. For example, Descartes rule of signs may be used. If one has a problem in transient phenomena the Routh† stability criterion may be employed. If $n \leqq 5$ in (1), then

* Presented to the Illinois Section of the Association, May 24, 1947.

† H. S. Wall, Polynomials whose zeros have negative real parts, this MONTHLY, vol. 52, No. 6, 1945, pp. 308–332.

synthetic division to find real roots may be used at once.

We shall employ the term "large" in this paper. This is a purely relative term of comparison between the given quantity said to be large and other quantities with which it is being compared. In some problems the given quantity is large if it is at least 10 times each of the other quantities involved; sometimes the factor is at least 100. The concept of "large" is used to throw away terms, and what terms may be thrown away depends on the errors one is allowed to make in the problem. When we say that a real root is *numerically large* we mean that this root is numerically so large compared to the remaining roots we can simplify the algebraic expressions of this paper by keeping only the terms of highest power in the numerically large root. To simplify algebraic expressions we shall also refer to complex roots with numerically large real or imaginary parts. Here we compare the numerically large part with the other part of the root, the absolute values of the remaining roots, and the quantities $A_1, \cdots, A_n$.

If there are numerically large real roots of (1), they should be found by the methods of Section 2. By the process of Section 3 the complex roots should be located.

In the case of a stable transient system the roots with numerically large real or imaginary parts may be discarded.† G. A. Philbrick‡ has developed an analyzer of physical systems based on the idea that the transient performance is determined by only a few of the roots of the corresponding equation (1), so that physical systems can ordinarily be described by differential equations of low order.

2. Numerically large real roots. In the solution of (1) it is advisable first to find approximately the numerically large real roots, and to use the reduced equation after each such root is found. Probability considerations show that an advantage in removing a numerically large real root is that the reduced equation often has relatively small coefficients compared to the coefficients in (1).

We shall explain later how to test the closeness of an approximation to a numerically large real root of (1). We note here that if (1) has a numerically large real root, the equation

$$x - a_1 = 0 \tag{5}$$

might give a good approximation to α_1. It is the author's experience that this approximation generally is not good.

It will now be proved, as is well known, that a numerically large real root α_1 of (1) approximately satisfies the equation

$$x^2 + a_1 x + a_2 = 0. \tag{6}$$

† Discarding roots with numerically large real or imaginary parts has been successful in the author's experience except in the case of gas turbines where numerically large real roots of an equation (1) correspond to the temperature in the combustion chamber which can vary rapidly, and the essentially smaller roots correspond to the speed of the wheel which changes relatively slowly.

‡ Presented to the SAE National Aeronautics Meeting, Oct. 2–4, 1947, Los Angeles, California, in a paper entitled, Electronic analog studies for turbo-prop control systems.

Let σ_1 be the sum of the remaining roots $\alpha_2, \alpha_3, \cdots, \alpha_n$ of (1). It follows that $a_1 = -(\alpha_1+\sigma_1)$. Let σ_2 be the sum of the products of $\alpha_2, \alpha_3, \cdots, \alpha_n$ in pairs. Then if $|\alpha_1|$ is large enough relative to $|\alpha_2|, \cdots, |\alpha_n|$ the quantity σ_2 is small compared to $\alpha_1\sigma_1$. Therefore a_2 is approximately $\alpha_1\sigma_1$. Then (6) is approximately $x^2-(\alpha_1+\sigma_1)x+\alpha_1\sigma_1=0$. Obviously α_1 satisfies (6) approximately.

There are equations such that (6) does not give a good approximation, but

$$x^3 + a_1x^2 + a_2x + a_3 = 0 \tag{7}$$

determines α_1 to sufficient accuracy. If (5), (6), or (7) do not give α_1 to sufficient accuracy we may form similar equations using more coefficients of (1). In the author's experience this has not been necessary.

The author has not found the usual synthetic division process satisfactory for finding numerically large real roots. As an example consider the synthetic division

				$\vert -101$
1	121	2030	1010	
	−101	−2020	−1010	
1	20	10	0	

which shows that -101 is a root of a certain cubic. A computer would ordinarily not guess the exact value -101. Suppose that in attempting to find the root -101 he tries -90. Ordinary synthetic division follows.

				$\vert -90$
1	121	2030	1010	
	−90	−2790	68,400	
1	31	−760	69,410	

Since the remainder 69,410 is large compared to the entry 1010, he would be inclined to think that he is very far from the actual root, when he is really quite near.

The synthetic division used above is left to right synthetic division. Suppose now that the computer tries right to left synthetic division with the value -90. Keeping one decimal place he obtains the following.

				$\vert -90$
1	121	2030	1010	
		−2018.8		
	22.4	11.2		

The remainder 11.2 is obtained by dividing -1010 by -90, the entry -2018.8 is 11.2 minus 2030, and the first remainder 22.4 is the result of dividing -2018.8 by -90. Since $90+22.4$ is near 121 the computer knows that -90 is near a root.

It is the author's experience that if one makes a guess $\alpha_1+\epsilon$ to a numerically large real root α_1 of (1), even a small error ϵ will yield a large remainder when left to right synthetic division is used, and it is necessary for the computer to work with large entries compared with the remainder and the entries obtained

in right to left synthetic division. It is a simple matter to justify these conclusions on purely mathematical considerations. For the sake of brevity these will be omitted.

Probability considerations show that there is a good chance that equation (1) will have a real root which dominates the others in absolute value.*

The preceding discussion will now be illustrated with the equation

$$x^6 + 228x^5 + 6{,}000x^4 + 37{,}000x^3 + 210{,}000x^2 + 320{,}000x + 274{,}000 = 0 \tag{11}$$

which arose in the study of a new diesel electric locomotive. The actual process used in the industrial solution will be given. Since the coefficient of x^5 is large compared to the leading coefficient it appears that there is probably a real root with large numerical value. Solution of (6) yields the approximation $\alpha_1 = -198$. In the following synthetic division of the left member of (11) by -198 we start at the right and round off numbers to convenient values.

1	228	6,000	37,000	210,000	320,000	273,000	$\vert -198$
		−5,800	−36,000	−208,000	−319,000	−273,000	
	29	182	1,050	1,610	1,380	0	

The fifth remainder 1,380 is obtained by dividing 274,000 by 198, the fourth by dividing 319,000 by 198, and so on. The first remainder is approximately 29 instead of the required value, $a_1+(-198)$, that is, 30. We therefore try -199 as shown.

1	228	6,000	37,000	210,000	320,000	273,000	$\vert -199$
		−5,800	−36,000	−208,000	−319,000	−273,000	
	29	181	1,040	1,600	1,370	0	

The first remainder is 29; also here $a_1+(-199)$ is 29. Therefore -199 is the more satisfactory approximation to the large root of (11) under consideration.

We now proceed to the reduced equation

$$x^5 + 29x^4 + 181x^3 + 1{,}040x^2 + 1{,}600x + 1{,}370 = 0 \tag{12}$$

for which the root -23 is readily found by the method just described.

3. Complex roots. There will now be described a modified synthetic division process which ordinarily leads rather rapidly† to complex roots of (1), in contrast to standard techniques in common use. Conjugate complex roots of (1) determine a quadratic factor of the given polynomial on the left of (1). There-

* Several thousand equations of type (1) arising in practical problems were solved. These represented rather well all of the major fields of engineering. In almost every case there was a real root numerically large compared to the other roots, and it was readily found by synthetic division performed from right to left.

† Sixth and eighth degree equations which would have taken a day by Graeffe's method have always been solved by the author in less than 30 minutes for two or three significant digits.

fore division of this polynomial by a general quadratic function, x^2+ex+f, will be explained. The forms of the remainder in this division will be established by induction.

Let r_i, s_i denote the pair of remainders after the ith division. Therefore, clearly,

$$r_1 = a_1 - e, \qquad s_1 = a_2 - f,$$
$$r_2 = e^2 - f - a_1e + a_2, \qquad s_2 = ef - a_1f + a_3,$$
$$r_3 = -e^3 + 2ef + a_1e^2 - a_2e - a_1f + a_3, \qquad s_3 = -e^2f + f^2 + a_1ef - a_2f + a_4.$$

These results suggest that the remainders after the ith division have the form

$$r_i = (-1)^i[e^i - (i-1)e^{i-2}f + \cdots], \tag{13}$$
$$s_i = (-1)^i[e^{i-1}f - (i-2)e^{i-3}f^2 + \cdots],$$

in which all terms in the a's and terms of lower degree in e than those exhibited have been omitted. The next row in the division will be

$$r_i, \qquad er_i, \qquad fr_i.$$

Therefore

$$r_{i+1} = s_i - er_i, \qquad s_{i+1} = a_{i+2} - fr_i.$$

Hence

$$r_{i+1} = (-1)^{i+1}[e^{i+1} - ie^{i-1}f + \cdots],$$
$$s_{i+1} = (-1)^{i+1}[e^if - (i-1)e^{i-2}f^2 + \cdots],$$

in which all terms in the a's and terms of lower degree than those shown have been dropped. Since the formulas (13) have been verified when $i=2$ (as well as 3), it follows by induction that they are valid if $i=2, \cdots, n-1$.

It will now be assumed that $f\neq 0$. It follows from (13) that, as polynomials in e, one of r_{n-1}, s_{n-1} is of odd degree in e. Now a real polynomial equation of odd degree has a real root. Therefore, if $f\neq 0$, there is an e such that one of the final remainders r_{n-1}, s_{n-1} vanishes. Such a root of $r_{n-1}=0$ or $s_{n-1}=0$ can be found by studying the sign of the polynomial. There may be such a root when $f=0$.

It will now be assumed that there are numbers e, f such that $s_{n-1}(e, f)=0$. In the zw-plane

$$s_{n-1}(z, w) = 0 \tag{14}$$

is a curve, and (e, f) is a point on this curve. If there is a point (e, f) such that (e, f) is on (14) and also on

$$r_{n-1}(z, w) = 0, \tag{15}$$

then an approximation (e_1, f_1) to (e, f) may be found by trial.

Thus if for a given value of e, one of the remainders $r_{n-1}(e, f)$, $s_{n-1}(e, f)$ is of odd degree in f, we can find a point (e, f) on one of the curves (14), or (15) by trial of values of f.

Unless the problem is a particularly difficult one, it is possible without studying the signs of the polynomials $r_{n-1}(e, f)$, $s_{n-1}(e, f)$ to find rather quickly a point (e_1, f_1) near a point of intersection (e, f) of the curves (14) and (15).

It should be noted that either or both of the curves $r_{n-1}(z, w) = 0$ or $s_{n-1}(z, w) = 0$ may have more than one branch. For example if $n = 4$, then $r_3(z, w) = 0$ is a cubic curve with $z = a_1/2$ as asymptote. All branches of (14) and (15) should be tested to find a point (e_1, f_1) which approximately satisfies (14) and (15).

We return to the locomotive example of Section 2. After the factors corresponding to the roots -199 and -23 of the locomotive equation (11) have been removed, the reduced equation

$$x^4 + 6x^3 + 42x^2 + 67x + 60 = 0 \tag{16}$$

is obtained. Equation (16) will be solved by the methods described above. Examination of (16) with trial divisors shows that synthetic division by a real number yields a positive remainder so that there are no real roots of (16). The Routh test shows that the real parts of the roots are negative so that the numerical value of such a real part cannot exceed 6 (*i.e.* minus the sum of the roots). As a first guess we try $(e_1, f_1) = (2, 0)$. The table for division is

		1	4	34	
2, 0	1	6	42	67	60
		2	0		
		4	42		
			8	0	
			34	67	
				68	0
				−1	60

The quotient is $x^2+4x+34$, and the remainder $-x+60$.

If $e_1 = 1.9$ instead of 2, the remainder $2x+60$ ($r_3 = 2$, $s_3 = 60$) is obtained. Since $f = 0$ in each case, and since $r_3(2, 0) < 0$, $r_3(1.9, 0) > 0$, there is a point $(e_1, 0)$ on the curve (15) for e_1 between 1.9 and 2.

We shall now show how a pair of values (e_1, f_1) is chosen so that (e_1, f_1) is approximately on both curves (14) and (15). If $(e_1, f_1) = (2, 1)$ the table at the top of the following page is obtained. Thus s_3 has decreased from 60 to 27. Next from $(e_1, f_1) = (2, 2)$ the remainder $-5x-4$ is obtained. Thus $r_3(2, 2) = -5$, and $s_3(2, 2) = -4$. Since these numbers are both small it is true that x^2+2x+2 is an approximation to a quadratic factor of the left side of (16). Since the remainders r_3 and s_3 are both negative a new approximation to (e_1, f_1) would be (1.8, 1.8). The remainder is now $0.8x+1.4$, and the quotient is $x^2+4.2x+32.6$.

Further trial with neighboring values shows that for two-digit accuracy the expression $x^2+1.8x+1.8$ may be used as an approximation to a factor of the left side of (16).

$$
\begin{array}{rcccccc}
 & & 1 & 4 & 33 & & \\
\cline{2-6}
2,1 & 1 & 6 & 42 & 67 & 60 & \\
 & & 2 & 1 & & & \\
\cline{3-4}
 & & 4 & 41 & & & \\
 & & & 8 & 4 & & \\
\cline{3-5}
 & & & 33 & 63 & & \\
 & & & & 66 & 33 & \\
\cline{4-5}
 & & & & -3 & 27 &
\end{array}
$$

Approximate roots of the locomotive equation (11) are therefore

$$-199, \quad -23, \quad -2.1 \pm 5.3i, \quad -0.9 \pm i. \tag{17}$$

4. Simplification of solutions for transients. The approximate solution of the differential equation (2) corresponding to the algebraic equation (11) is of the form

$$\begin{aligned} y = {} & B_1e^{-199t} + B_2e^{-23t} + e^{-2.1t}(C_3 \cos 5.3t + C_4 \sin 5.3t) \\ & + e^{-0.9t}(C_5 \cos t + C_6 \sin t). \end{aligned} \tag{18}$$

We shall show that if t represents time (as it does in this example) we are justified in discarding the first two terms on the right of (18).

We consider a solution (4) of (2) and (3) in which $n \geqq 2$, α_1 is real and the solution is stable (the real parts of $\alpha_1, \cdots, \alpha_n$ are negative). The B's satisfy the equations

$$\sum_{i=1}^{n} B_i = A_1, \quad \sum_{i=1}^{n} \alpha_i B_i = A_2, \cdots, \quad \sum_{i=1}^{n} \alpha_i^{n-1} B_i = A_n. \tag{19}$$

If Δ is defined as the determinant of the coefficients of the B's in (19), then $\Delta \neq 0$ because Δ is a product of the differences $\alpha_i - \alpha_j$ for $i \neq j$.* We note that Δ is the determinant in the denominators of the solution of (19) for $B_1, \cdots, B_n$. The determinant Δ is of degree $n-1$ in α_1, whereas α_1 does not occur in the determinant which is the numerator of B_1. This is not generally true for the numerators of $B_2, \cdots, B_n$. It follows that if α_1 is sufficiently large numerically (compared to $A_1, \cdots, A_n$ as well as the other roots), then the coefficient B_1 in (4) is generally small in comparison with the other B's. Hence as t increases the term $B_1e^{\alpha_1 t}$ decreases to zero rapidly compared to the other terms, and may be dropped from the solution. We remark that for transients we are concerned with what happens for increasing t.

* L. Weisner, Introduction to the Theory of Equations. Macmillan, 1938. pp. 55, 56.

If the roots α_1 and α_2 are conjugate complex roots of (1), the notation $u+vi$ will be used for α_1. The numerators of B_1 and B_2 corresponding to these roots are of lower degree in v than the determinant Δ. If v is large compared to $|u|$, the absolute values of the remaining roots, and $A_1, \cdots, A_n$, the quantities B_1 and B_2 are in general relatively small compared to the remaining B's. Therefore if $n \geqq 3$, and v is large enough in the above sense, and further $|u|$ is not small numerically compared to the real parts of the other roots of (1), the first two terms of (4) can be discarded. The same is true if $|u|$ is large compared to v, the absolute values of the remaining roots, and $A_1, \cdots, A_n$.

TWO TEASERS FOR YOUR FRIENDS

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You may test the native mathematical ability and the mathematical training of your friends with the following variations of a very old problem. Our first problem has a simple but significant catch to it which a bright ten-year old may discover at once, but which most of your non-mathematical colleagues will miss; the second has a catch for the professional mathematician. Try them on your friends. We present the problems as they might arise at a dinner table.

PROBLEM 1. I am sitting at a table four feet wide. I place my knife at one edge of the table, then half way across the table, then half way from there to the other edge, then half way from there toward that edge, and so on indefinitely. How long a time is it before the knife is at that edge? We understand that we are talking about a mathematical knife having no width, that the measurements indicated at the various steps are mathematically exact, and that the problem is strictly mathematical.

Of course most of your friends will tell you that the knife never reaches that far edge. But your bright ten-year-old may say that there was nothing about time in the data and hence he can reach no conclusion about how long a time it would take. Your colleagues may be taken aback by this remark.

PROBLEM 2. Add to the data of Problem 1 the statement that it took one second for the knife to be moved in the first step, a half a second for the second step, a fourth of a second for the third step, and so on, with the general rule that it took $1/2^{n-1}$ seconds for the nth step, where n is a positive integer. Now what is the answer to the question of Problem 1?

This goes to the heart of a very old problem. Most of your friends, after a little thought, will say that the proper answer is *two seconds*. What do you say? My answer is given on page 377.

A MACHINE FOR PLAYING THE GAME NIM

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1. Introduction. The game Nim is played as follows. Starting with an arbitrary number of objects arranged in separate piles, two persons play alternately, removing one or more objects from any one pile in a given turn. The person taking the last object wins, for the case which we have designated as the *normal case*, but loses in the *reversed case*. An interesting feature of the game is that there is a complete mathematical theory for it, by which a player can infallibly win if given a choice of first or second move. This theory is of such a nature that the computations required can be carried out by simple electrical circuits. It is the purpose of the present article to describe one method of doing so. To prevent the problem from becoming too trivial, we shall add the restriction that no relays or other devices are allowed; that is, the part of the circuit concerned with computation can contain ordinary switches only. In the early stages of the design this was a somewhat annoying limitation; but with the design completed, one finds, after all, that introduction of relays will not lead to any obvious simplification.

Nim was originally played with three piles. In this form it is said to have had its origin in the Orient, perhaps in China, and to be of great antiquity. In 1906 the theory for an arbitrary number of piles was published [1]. This theory, which also takes account of the reversed case, is summarized in Section 2 of the present article. In [2] is described a more general game, in which one can remove objects from k piles in each turn, rather than from only one pile. The theory is given for the normal case. In 1940, E. U. Condon* obtained a patent on a machine for playing the normal case of the game forming the subject of this article [3]. The circuit appears to be quite different, however, from that considered here, and makes extensive use of relays. A model of this machine for four piles with a maximum of seven objects in each was actually built as an exhibit for the New York World's Fair. Since that machine contained over a ton of equipment, while the present one weighs only about five pounds, it is felt that this article will be of interest.

In 1941, a machine was designed by the author for playing the normal and reversed cases with $k=1$. Relays were used, but otherwise the machine was equivalent in principle to that of the present article. In 1941–42 the author extended the theory to include the reversed case for arbitrary k, and proved that the conditions previously employed, if slightly modified, are both necessary and sufficient for correct moves. These elementary results were preliminary to the design of a machine for playing the normal and reversed cases with both k and n arbitrary. The design of such a machine was completed in 1941–42 [4]. It may be noted that the two cases, $k=1$ and $k=k$, differ only trivially as far as the theory is concerned; but, by this slight increase in generality, the problem of mechanization is greatly complicated. Here we assume $k=1$, with no restriction on n, the number of piles.

* Now head of the National Bureau of Standards.

2. Mathematical theory. Although the theory of play is well known, it must be reviewed here, since it forms the basis for the proposed design. First, one must count the objects in each pile and write these numbers in the binary system. For example, if there are four piles containing 7, 12, 10 and 4 objects, respectively, one would write down the set of numbers

$$\begin{array}{r} 111 \\ 1100 \\ 1010 \\ 100. \end{array} \tag{1}$$

Next, add corresponding digits, starting with those at the right. The sum of the unit digits is 1 in the present case; the sum of the two's digits is 2, the sum of the four's digits is 3, and the sum of the extreme left-hand digits is 2. If these sums of corresponding digits are all even, then the position is said to be a safe position, but if one or more of these sums is odd, the position is not safe. The position shown, for example, is not safe.

Starting with a position which is not safe, one can always leave a safe position by playing on one pile only. This operation, which is of importance in the sequel, is carried out as follows. First, one must pick the leftmost column of all those columns which gives an odd sum. In our example, the column in question would be the second column from the left, that is, the column for which the sum is 3. Since the sum of digits in this column is odd, at least one of the digits in it must be a 1. Pick any pile which gives a 1 in this position; for example, the first, second, or fourth piles would be permissible in our special case. Change the 1 in the *odd farthest left* (as we shall call the column in question) to a 0, and adjust the remaining digits in such a way that the other sums are all even. The new binary number thus obtained represents the number remaining in that pile after the completion of the move. The number of objects is actually decreased as required. Also the resulting position is safe.

We have shown that a position which is not safe can always be turned into a safe position by legitimate operations. On the other hand, the reader will easily verify that a person confronted with a safe position can never leave a safe position by moving, as required, in one pile only. Given the proper initial conditions, it follows that a player can always leave his opponent confronted with a safe position, and that he himself will never be confronted with such a position. Since the number of objects continually decreases, eventually the position must be reached for which all piles are empty. This position is safe, however, and hence it will be the opponent's turn to play when it arises.

The theory as described above is for the case in which the object of the game is to make the last move. Because the theory is slightly simpler, this case has been termed the *normal case*. One may also play a game, which we have designated as the *reversed case*, in which one's purpose is to force the opponent to take the last object. Only a trivial modification of the theory is required for this case, namely, to leave an odd instead of an even number when the game has so

far progressed that just one pile has more than one object. An opportunity for this modification will always arise, since the position in question must occur eventually, and is not safe. This reversed case, which leads to a trivial modification of the theory, can be dealt with by an equally trivial modification of the circuit; and hence the machine is designed to play that form of the game as well as the other.

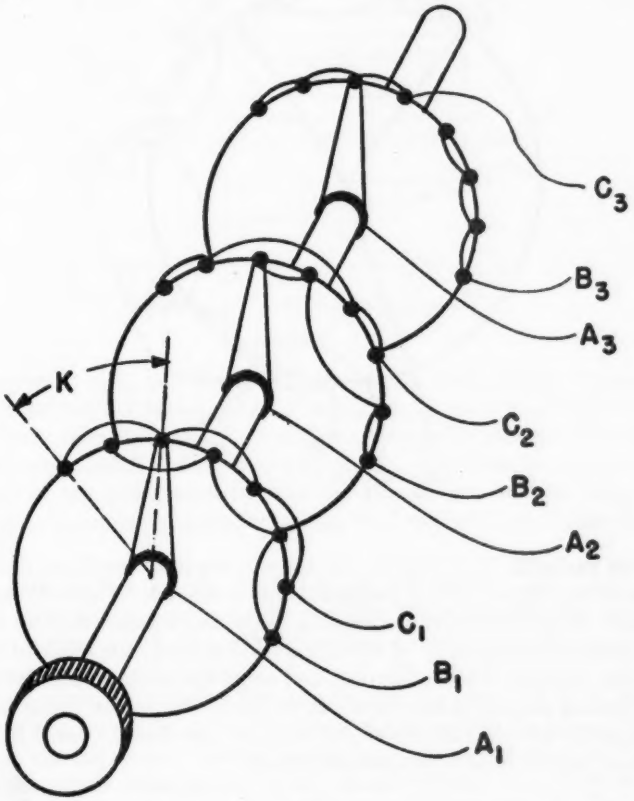

FIG. 1a

3. Conversion of numbers to the binary system. Our first task is to convert the number of objects in each pile to the binary system. Let us suppose that these numbers are introduced by changing the contact position on a rotary switch of conventional design, and that the switch is supplied with as many layers of contacts ("pies") as there are digits in the binary number required for representing the maximum number to be put in that pile. The connections for pies representing the units', two's, and four's digits are then as shown in Figure

1a, and the general arrangement for the jth pie, which gives the jth digit, is shown in Figure 1b. A rigorous proof that the circuit has the properties described can be given.

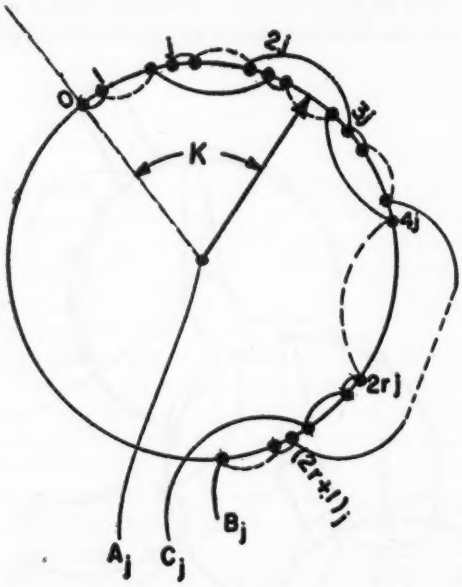

FIG. 1b

4. Even and odd. According to the theory, the next step is to find the sum of corresponding digits. Such a problem is quite difficult to solve by any simple arrangement of switches, and we inquire, therefore, whether finding the sum is really necessary. In the course of the calculation the only use made of the sum is to determine whether it is even or odd, and hence the problem may be revised as follows: Given a series of n switches (one for each pile) to determine whether an even number or an odd number of the switches are closed. Stated in this way, the problem can be solved by a simple circuit.

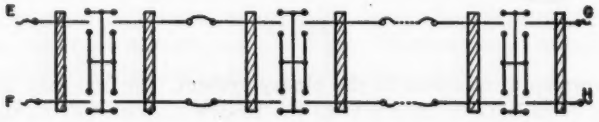

FIG. 2

Let us notice that the evenness or oddness is reversed when any one switch of the series reverses its position. Hence the original problem, to determine whether a sum is even or odd, now becomes the simpler problem of arranging switches to

reverse connections in this way. A solution is given in Figure 2. The switches are double-pole, double-throw, one switch being required for the jth digit in each pile. It is easy to verify that changing the position of any one of these switches has the effect of interchanging the terminal wires G, H; in other words if G was

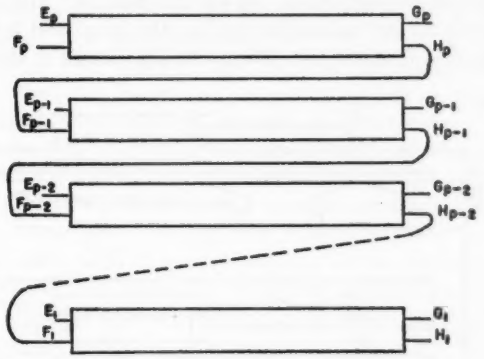

FIG. 3

initially connected to E, and H to F, then changing the position of a switch will leave the circuit with G connected to F, and H to E. Also, if all switches are in the up position, then E, G and F, H will be connected. It follows that E and G will be connected, and F and H will be connected, whenever an even number of switches are in the down position; but the connections will be interchanged whenever an odd number of switches are in the down position. Hence the problem is solved.

5. Odd farthest left. By the foregoing circuits we can determine whether the sums of corresponding digits are even or odd. We must next pick out the odd sum that is farthest left. Such a selection may be made by combining the separate circuits as indicated in Figure 3. From the discussion of Figure 2, we know that the wires H_j in Figure 3 will be neutral or connected to F_j, according as the sum of the jth digits is even or odd, and the opposite is true of the wires G_j. Hence, in the circuit as a whole, a given wire will be connected to F_p if and only if j is the largest subscript for which the sum is odd. For the wires F_j will be connected to F_p if and only if all the sums for greater j are even; and G_j will be connected to F_j if and only if the sum for that j is itself odd.

We remark in passing that the wires E_j for $j>1$ are not really used, and E_1 itself is needed only for the reversed case. Hence the first double-pole switch shown in Figure 2 may generally be replaced by a single-pole switch. This simplification, which is made in Figure 4, is not made in Figure 2 because the circuit (for $j=1$) will be required later in the complete form.

6. Completion of machine for normal case. Having selected the proper column in the general array corresponding to (1) we are required next to select the

piles for which the binary numbers have a 1 in this column; in other words, we must select the proper rows. To this end let us add an extra set of switches which will be on if the jth digit is 1, off otherwise. That such switches can be constructed follows from Section 3. If these switches are connected in series with the previous arrangement for finding the odd farthest left, and the whole is connected in series with an indicating lamp, then the lamp will light if and only if there is a 1 in the odd farthest left. The complete circuit is shown in Figure 4.

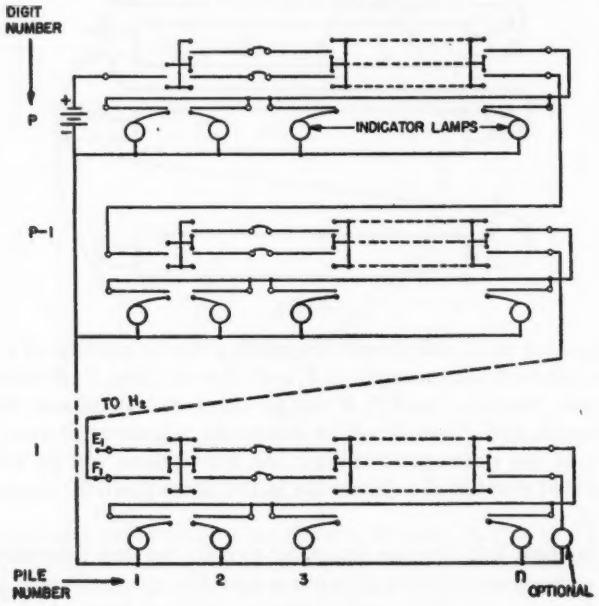

FIG. 4

For a given pile there will be several lamps, one for each digit in the binary representation, and these lamps cannot be replaced by a single one without upsetting the operation of the circuit. However, they may be arranged in a compact group, so that they function like a single lamp as far as indicating purposes are concerned. Introduction of relays at this stage would eliminate the need for using separate lamps in this way, but it is not evident that any advantage would be gained. The light labeled "optional" in Figure 4 goes on whenever it is the opponent's move. Although not really necessary in the present context, the possibility of such a connection would be useful if mechanization of the machine, for automatic operation, were undertaken.

An experimental machine has been designed for a maximum of seven objects in each of four piles. Hence there are four switches, and each switch has eight positions. Since any number not exceeding 7 can be written in the binary system with three digits, there are three indicator lights for each pile. These are

grouped behind a translucent screen covering a large round hole above each switch. Neon bulbs are used partly because of their low current requirement and partly because they will not light if two are in series. The input is 110 volts, a.c.

To use the machine, one specifies the initial position by adjusting the switches. For example, if the four piles are to have 2, 3, 5, and 7 objects, respectively, then the switches would be set to 2, 3, 5 and 7. If the position is safe no lights will light, and it is the opponent's move. If, however, the position is not safe, which is true of our example, then the indicator lights will be on for those piles, and only those piles, in which a correct move can be made. In the special case cited, the lights over the first, second, and fourth switches would be on. The operator of the machine selects any one of these permissible piles and turns its switch to the left (that is, reduces the number of objects) until the light for that pile is off. It follows from the theory that all the other lights will be off, and thus the move is completed. The opponent makes his move by adjusting whichever switch he pleases, whereupon the operator of the machine repeats the procedure just described. This process is continued until the game is over.

It should be observed that the move is completely specified by the machine; the only contribution made by the operator is to select the pile he prefers, when a correct move can be made in more than one pile, and to make the adjustments indicated. No calculation on his part is required at any time. The operation of selecting the alternative, when several possibilities are presented, could be eliminated by a circuit similar to that of Figure 3; that is, one could so arrange matters that, for example, only the leftmost light of the series would go on. The present arrangement is believed preferable because it allows the machine to indicate all possible correct moves.

7. Concluding remarks. To make the machine play the reversed case, we arrange a circuit which moves the wire labeled "to H_2" in Figure 4 from F_1 to E_1, when it first happens that no pile has more than one object. In the experimental machine a switch is provided for distinguishing the normal and reversed cases.

It is believed that the number of switches used is the minimum for a design of this general type; the present machine has four switches with about 10 pies each and eight contacts in each pie. For the general case the number of pies is approximately proportional to $n \log m$ and the number of contacts on a given pie is $m+1$, if m is the maximum number of objects in a given pile.

References

1. Bouton, C. L., Nim, a Game with a Complete Mathematical Theory, Ann. Math. Sec. Ser. v. 3, No. 1, Oct. 1901.

2. Uspensky and Heaslet, Elementary Number Theory, McGraw-Hill, 1939.

3. Condon, E. U., Edgewood, Tawney, G. L., Derr, W. A., Wilkinsburg, Westinghouse Electric and Manufacturing Co., Pittsburgh, Pa., A Machine to Play the Game of Nim, U. S. Patent no. 2,215,544, Sept. 24, 1940.

4. Redheffer, R. M., A Machine for Playing Generalized Nim, B.S. Thesis in Mathematics, 1943, Mass. Inst. Tech.

FURTHER REMARKS ON THE APPROXIMATION OF NUMBERS AS SUMS OF RECIPROCALS

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1. Bibliographic notes. After publication of the author's recent paper on this subject,† several persons called his attention to the fact that some of his results were known previously. Beginning with their leads, a thorough search of the literature pertaining to R- and $\overline{R}$-expansions gave rise to the following bibliography which is believed to be exhaustive:

(a) O. Perron. *Irrationalzahlen.* Walter de Gruyter, Berlin; 2nd ed. 1939, 1st ed. 1921; pp. 116–122 for both editions. Discusses "Sylvester series" (in the 1921 edition they are called "Engel series of the second kind") which are essentially the same as R-expansions. But a "Sylvester series," in addition to including the R-expansions, also includes the cases where in $\sum 1/N_i$, beyond a certain value $i=i_0$, one always has $N_{i+1}=N_i^2-N_i+1$. Thus a "Sylvester series" yields an infinite series alternative to the terminating R-expansion for a rational number, where the last term of that R-expansion is $1/(N_{i_0}-1)$. Theorem 49 on p. 119 is essentially another statement of the author's Theorems I and II on p. 136.†

(b) J. J. Sylvester. On a Point in the Theory of Vulgar Fractions, *Amer. Jour. of Math.*, III, 1880, pp. 332–335; also, Postscript to Note on a Point in Vulgar Fractions, *ibid.*, pp. 388–389. Discusses "Sylvester series" (see (a)) or "sorites" (Sylvester). On p. 333 are statements (but without proofs) which are essentially the author's Theorems I and II.†

(c) G. Stratemeyer. Stammbruchentwickelungen für die Quadratwurzel aus einer rationalen Zahl, *Mathematische Zeitschrift*, XXXI, 1930, pp. 767–768. Expresses a quadratic irrationality by means of an R-expansion.

(d) G. Stratemeyer. Entwicklung positiver Zahlen nach Stammbrüchen, (Dissertation) *Mitteilungen des Mathematischen Seminars der Universität Giessen*, XX Heft, 1931, pp. 3–10, 25. Studies series of the form $\gamma=a+\sum_{\nu=1}^{\infty}\epsilon_\nu/q_\nu$, where the ϵ_ν's are $+1$ or -1. However this is not the $\overline{R}$-expansion, the discovery of which Stratemeyer seems to have missed; at no step is his sequence dictated by the idea of coming closest to the true value; in fact, those q_ν's satisfy $q_{\nu+1} \geqq q_\nu(q_\nu \pm 1)$, similar to the R-expansion. Stratemeyer obtains results essentially equivalent to the author's Theorems I and II, but in slightly more general form, involving the ϵ_ν's. There is included the derivation of his R-expansion for quadratic irrationalities, given in (c). Stratemeyer also gives the first five terms of the R-expansion for the decimal part of π.

(e) W. Sierpiński. Sur un algorithme pour développer les nombres réels en séries rapidement convergentes, *Bulletin International de l'Académie des Sciences*

* National Applied Mathematics Laboratories, National Bureau of Standards.

† The Approximation of Numbers as Sums of Reciprocals, this MONTHLY, March, 1947, pp. 135–142. The reader's attention is called to an obvious error at the bottom of p. 135, where $1/(a_i^2+a_i)$ should read $1/(a_i^2-a_i)$, and to an error on p. 139, 3rd line, where $1/(a_i^2-a_i)$ should read $1/(a_i^2+a_i)$. Also, note that the present article employs the notation $1/N_i$ in place of $1/a_i$ which was used previously for the ith partial fraction.

de Cracovie, Classe des Sciences Mathématiques et Naturelles. Série A, Sciences Mathématiques, 1911, pp. 113–117. Studies the third type of expansion into reciprocals mentioned on pp. 138, bottom, and 139, top, of the author's paper† (which was not labeled) and proves the same theorem concerning the irrationality of an expansion having an infinite number of terms. But Sierpiński states the closeness of the ith approximation correctly as $1/a_i(a_i+1)$ whereas the author† gave it incorrectly as $1/a_i(a_i-1)$. In actual magnitude, that is still about the same as the closeness of approximation of an R-expansion, since as long as i does not denote the last term, $a_i+1=a_i'$ where after $i-1$ terms, $1/a_i$ would be the continuation as a Sierpiński expansion and $1/a_i'$ the continuation as an R-expansion. (Of course, the preceding statement is not to be taken to mean that $a_i+1=a_i'$ for every a_i and a_i' in the R- and Sierpiński expansions respectively).

(f) F. Engel. *Verhandlungen der 52. Versammlung deutscher Philologen und Schulmänner*, Marburg, 1913. On pp. 190–191 is an abstract of his lecture on expansions into reciprocals. There is an error on p. 190 in the statement of the criterion of rationality, which should read $q_{n+v+1}=q_{n+v}^2-q_{n+v}+1$, (the $+1$ was omitted).

(g) J. H. Lambert. *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, Zweyter Theil, Berlin, 1770. Chapter III, Verwandlung der Brüche. On pp. 99–104 is a definition, with illustrations, of the $\overline{R}$-expansion, with a statement of the rapidity of convergence as "double the number of places" at each step; but no proofs of anything are given. The first three terms in the $\overline{R}$-expansion of the decimal part of π are given correctly, but the fourth is given

$$\text{incorrectly as } -\frac{1}{44338\ 40233\ 62059} \text{ instead of } +\frac{1}{15164\ 89608\ 87729}.$$

(h) O. Spiess. Über eine Klasse unendlicher Reihen, *Archiv der Mathematik und Physik*, Ser. 3, XII, 1907, pp. 124–134. Deals with R-expansions; contains a proof of Theorem II; develops R-expansions for certain quadratic irrationalities (see (c) and (d)).

(i) E. Cahen. Note sur un développement des quantités numériques, qui présente quelque analogie avec celui en fractions continues, *Nouvelles Annales de Mathématiques*, Ser. 3, X, 1891, pp. 508–514. Treats R- and Sierpiński expansions (see (e)), the latter twenty years before Sierpiński. Hence if a single name should be given to the hitherto-unnamed third type of expansion, it is "Cahen-expansion." Proves finiteness of expansions for rational numbers, gives inequalities for remainders, and states a theorem concerning uniqueness similar to Theorem I.

2. Comparison with simple continued fractions. It is of interest to compare the number-theoretic aspects of R-expansions with that of the Euclidean algorithm which arises from the development of the fraction a/b into a simple

† *loc. cit.*

continued fraction. Thus the Euclidean algorithm arising from

$$\text{(A)}\qquad \frac{a}{b}=\frac{1}{M_1}+\frac{1}{M_2+\;\cdot\;\cdot\;\cdot+\frac{1}{M_m}}$$

can be expressed as

$$\text{(A')}\qquad \begin{aligned} b &= aM_1+r_1\\ a &= r_1M_2+r_2\\ r_1 &= r_2M_3+r_3\\ &\cdots\cdots\cdots\cdots\\ r_{m-3} &= r_{m-2}M_{m-1}+r_{m-1}\\ r_{m-2} &= r_{m-1}M_m. \end{aligned}$$

The r_i's are a decreasing set of positive integers and r_{m-1} is the well known greatest common divisor (g.c.d.).

Now consider the R-expansion

$$\text{(B)}\qquad \frac{a}{b}=\frac{1}{N_1}+\frac{1}{N_2}+\cdots+\frac{1}{N_n}$$

$$\frac{a}{b}-\frac{1}{N_1}=\frac{aN_1-b}{bN_1}=\frac{r_1}{bN_1},\qquad \text{where } r_1<a.$$

Repetition of this argument on r_1/bN_1 in place of a/b leads to

$$\frac{r_1}{bN_1}=\frac{1}{N_2}+\frac{r_2}{bN_1N_2},\qquad r_2<r_1,$$

$$\frac{r_2}{bN_1N_2}=\frac{1}{N_3}+\frac{r_3}{bN_1N_2N_3},\qquad r_3<r_2,$$

$$\cdots\cdots\cdots\cdots$$

until there is reached

$$\frac{r_{n-2}}{bN_1N_2\cdots N_{n-2}}=\frac{1}{N_{n-1}}+\frac{r_{n-1}}{bN_1N_2\cdots N_{n-1}},\qquad r_{n-1}<r_{n-2},$$

$$\frac{r_{n-1}}{bN_1\cdots N_{n-1}}=\frac{1}{N_n}+0.$$

The above system can be expressed as

$$
\begin{aligned}
aN_1 &= b + r_1 \\
r_1N_2 &= bN_1 + r_2 \\
&\cdots\cdots\cdots\cdots \\
r_{n-2}N_{n-1} &= bN_1N_2 \cdots N_{n-2} + r_{n-1} \\
r_{n-1}N_n &= bN_1N_2 \cdots N_{n-1}.
\end{aligned} \tag{B'}
$$

Inspection of the foregoing system reveals that any r_i is a multiple of the g.c.d. of a and b, and this is noteworthy for r_{n-1}. Hence $r_{n-1}=1$ is sufficient but not necessary for a and b to be relatively prime. When $r_{n-1}=1$, the last denominator is $bN_1N_2 \cdots N_{n-1}$, and conversely. Thus in the R-expansion of a/b, $N_n=bN_1N_2 \cdots N_{n-1}$ is a sufficient but not necessary condition for a and b to be relatively prime.

A practical result of r_{n-1} being a multiple of the g.c.d., since in both the above and the Euclidean algorithm $r_i<r_{i-1}$, is that as a rule, for a/b fewer partial fractions are required in an R-expansion than in the simple continued fraction. For example,

$$
\frac{17}{90} = \frac{1}{6} + \frac{1}{45}, \quad \text{but} \quad \frac{17}{90} = \cfrac{1}{5 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2}}}}.
$$

Of course, there could be exceptions where the r_i's in the Euclidean algorithm decrease so rapidly as to reach in fewer steps a lower last value (which is the g.c.d.) than r_{n-1} in the R-expansion.

3. Closeness of approximation of R-expansions.

LEMMA 1. *For $i>1$, $N_i>N_1 \cdots N_{i-1}$, where $1/N_i$ is the ith term in the R-expansion.*

In (B'), since $b>a$,

$$
N_i = \frac{a+I}{a-I(i)} N_1 \cdots N_{i-1} + \theta(i), \tag{1}
$$

where $I>0$, $I(i)>0$, $0 \leqq \theta(i) < 1$. The right side of (1) is greater than

$$
\left(\frac{a}{a-1}\right) N_1 \cdots N_{i-1} + \theta(i) > N_1 \cdots N_{i-1}.
$$

(The converse is obviously not true, namely in $\sum 1/N_i$, if $N_i>N_1 \cdots N_{i-1}$, that does not imply that $\sum 1/N_i$ is an R-expansion. For example, consider $1/4+1/5+1/24$.) From Lemma 1 there readily follows the result stated by the

author without proof in the earlier article, namely.

THEOREM IV. *If p/q is an approximation to x obtained by the R-expansion, then the remainder $x-p/q$ is less than $1/q$.*

Proof. Let p/q arise from the first n terms of the R-expansion, where

$$x = \frac{1}{N_1} + \frac{1}{N_2} + \cdots + \frac{1}{N_n} + \text{remainder}. \tag{2}$$

The remainder is less than $1/(N_n^2 - N_n)$. (The argument is strengthened if p/q is reduced to lower terms since $1/q' > 1/q$ when $q' < q$; so consider the unreduced case $q = N_n N_{n-1} \cdots N_1$.) For $n=1$, the theorem is obvious. For $n>1$, Lemma 1 leads to

$$N_n - 1 \geqq N_{n-1} \cdots N_1, \tag{3}$$

from which

$$q = N_n N_{n-1} \cdots N_1 \leqq N_n(N_n - 1) = N_n^2 - N_n. \tag{4}$$

Hence

$$\frac{1}{q} \geqq \frac{1}{N_n - N_n} > \text{remainder}. \tag{5}$$

That $x-p/q<1/q$ is a best possible result is obvious from

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \cdots . \tag{C}$$

For suppose that there were some $\theta<1$ such that $x-p/q$ were always less than θ/q. Let $L(m)$ be a number arbitrarily close to 1, obtained by taking enough terms, that is, m in number, on the right side of (C), and then stopping, so that there are a definite number of terms in some R-expansion. (Note that (C) taken as an infinite series is not an R-expansion, even though it is a "Sylvester series"). Then for $n=2$, $L(m)-5/6$ would have to be less than $\theta/6$ for a fixed θ less than 1, which is impossible.

4. Closeness of approximation of $\overline{\text{R}}$-expansions.

THEOREM V. *If p/q is an approximation to x obtained by the $\overline{R}$-expansion, then*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q}.$$

except when $x=3/4$, when the $\leqq$ relation may hold.

It is instructive, at the price of redundancy, to prove two versions of this theorem, first a weaker version establishing the $\leqq 1/2q$ relation for all approximations p/q, and subsequently the stronger version yielding the sharper $<1/2q$ relation in all but one case.

LEMMA 2 (for weak version). *For $i>1$, $N_i>N_1 \cdots N_{i-1}$, where $1/N_i$ is the absolute value of the ith term in the $\overline{R}$-expansion.*

If the $\overline{R}$-expansion begins with 1, this first term may be deleted without affecting the following argument, because of the obvious facts that (a) if $1-1/N_1 \pm \cdots$ is an $\overline{R}$-expansion of x, then $1/N_1 \mp \cdots$ is an $\overline{R}$-expansion of $1-x$, and (b) if the original $\overline{R}$-expansion begins with 1, proving $N_i>N_1 \cdots N_{i-1}$ for the deleted expression proves it for the original expression by shift of index so that $N_1=1$ and the new N_i becomes the original N_{i+1}. Thus one need consider an $\overline{R}$-expansion beginning with $\frac{1}{2}$ or less. In the $\overline{R}$-expansion, let all terms $\pm 1/N_i$ be considered with only $+$ signs. From $1/N_1 \leqq 1/2$, and from $N_{i+1} \geqq 2(N_i^2-N_i)$, each $N_{i+1}=N_i^2-N_i+\epsilon_i$, $\epsilon_i>1$. Then, by the uniqueness theorem for R-expansions, the $1/N_i$'s are the partial fractions belonging to some R-expansion and must satisfy $N_i>N_1 \cdots N_{i-1}$, by Lemma 1.

Now consider the first n terms of the $\overline{R}$-expansion. For $n=1$,

$$\left| x - \frac{p}{q} \right| \leqq \frac{1}{2q}.$$

For if the $\overline{R}$-expansion commences with 1, the remainder $<1/2$ in absolute value; if it begins with $1/2$ followed by $\mp 1/4$, the next term must be $\pm$, so that

$$\left| x - \frac{1}{2} \right| < \frac{1}{4},$$

whereas if there is no third term, x must be $1/2+1/4$, or

$$\left| x - \frac{1}{2} \right| \leqq \frac{1}{4};$$

if it begins with $1/3$ (or $1/I$, $I>3$), the next term $\leqq 1/12$ (or $\leqq 1/2(I^2-I)$), so that

$$\left| x - \frac{1}{3} \right| < \frac{1}{6} \left(\text{or} < \frac{1}{2I} \right).$$

For $n>1$, by Lemma 2 and equations (3), (4) and (5) of Section 3, together with the inequality for the remainder after n terms in an $\overline{R}$-expansion, namely,

$$\left| x - \frac{p}{q} \right| \leqq \frac{1}{2(N_n^2 - N_n)}, \tag{6}$$

there follows

$$\left| x - \frac{p}{q} \right| \leqq \frac{1}{2q}, \tag{7}$$

which is the weakened version of Theorem 5.

The proof of Theorem V hinges upon the following inductive lemma.

LEMMA 3. *In an $\overline{R}$-expansion, whenever $N_i > N_1 \cdots N_{i-1}+1$, $i>1$, then also $N_{i+1} > N_1 \cdots N_i+1$.*

Proof.

$$N_{i+1} \geqq 2N_i^2 - 2N_i = N_i(2N_i - 2) > N_i\{2(N_1 \cdots N_{i-1})\}$$
$$= N_1 \cdots N_i + N_1 \cdots N_i > N_1 \cdots N_i + 1.$$

Now consider any $\overline{R}$-expansion. (a) If it begins with 1, the next term cannot be $-1/3$, so it is either $-1/4$ or $-1/(4+I)$. Since $4>1+1$, Lemma 3 applies to each succeeding approximation, and hence (3), (4), (5) and (7) hold with the $<$ sign in place of $\leqq$. (b) If it begins with $1/2$, the next term cannot be $\pm 1/3$ but may be $+1/4$. (If that is all, one has the exceptional case of $3/4$ where $\leqq$ holds, although when $3/4$ is written as the alternative $\overline{R}$-expansion $1-1/4$, there is no exception). If $1/4$ is followed by $- \cdots$ (cannot be $+ \cdots$) then $<$ applies at $1/2$. At $1/2+1/4$, the absolute value of the next term $\leqq 1/24$, so $<$ holds. Beyond the term $1/4$, since $4>2+1$, Lemma 3 is employed, and again (3), (4), (5) and (7) hold with $<$ signs. If it begins with $1/2$ followed by $\pm 1/(4+I)$, the same reasoning (Lemma 3, then (3), (4) and so on) again results in $<$. (c) If it begins with $1/3$ or $1/(3+I)$, the same line of reasoning holds; *e.g.* after $1/3$ there must follow $\pm 1/12$ or $\pm 1/(12+I)$, $|\text{error}| < 1/11 < 1/6$, then for $n>1$, Lemma 3 followed by (3), (4), and so on, to obtain $<1/2q$ in (7). Thus Theorem V is established.

That Theorem V contains the best possible inequality to hold for all $\overline{R}$-expansions, is apparent in considering the particular $\overline{R}$-expansion $1/2-1/4+1/I$. If instead of $<1/2q$, the closeness were expressed by $<1/(2+\delta)q$, for fixed δ, one would be led to a contradiction, since the I could be chosen sufficiently large so that the error after one term, where $q=2$, would be arbitrarily close to $1/4$.

It is of interest to compare the closeness of approximation of a fraction p/q obtained by either the R- or $\overline{R}$-expansion with a decimal approximation p'/q', where q' is a power of 10. On the basis of Theorems IV and V, there is a superficial resemblance between R-expansion approximations and "chopped" decimal approximations, as well as between $\overline{R}$-expansion approximations and "rounded" decimal approximations. But there is a fundamental difference between the use of successive decimal approximations to a given number, and the employment of either the R- or $\overline{R}$-expansion, which is more favorable to the latter. For after $\pm 1/N_n$, in the next partial fraction $\pm 1/N_{n+1}$, the N_{n+1} may be very much larger than the minimum value that could arise in an R- or $\overline{R}$-expansion, say about k times that minimum. In such cases, by obvious modifications of equations (3), (4), (5) and (7), based upon the strengthening of Lemma 3 with a factor of k on the righthand side of the inequality, it follows that not only the closeness of the nth but the closeness of all ensuing approximations p/q will be about $1/kq$ or $1/2kq$ for the R- or $\overline{R}$-expansion respectively.

MATHEMATICAL NOTES

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Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

ON THE TWELVE POINT SPHERE OF THE TETRAHEDRON

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Let G, O, O_{12} be the centroid, the circumcenter, and the center of the twelve point sphere* of an arbitrary tetrahedron $T \equiv ABCD$. Also let A_1, B_1, C_1, D_1 be the points (other than the vertices) where the altitudes AA', BB', CC', DD', cut the circumsphere (O). Let G_i $(i=a, b, c, d)$ denote the centroids of the tetrahedra $T_a \equiv A_1BCD$, $T_b \equiv B_1CDA$, $T_c \equiv C_1DAB$, $T_d \equiv D_1ABC$ and G_i' the centroids of the triangles BCD, CDA, DAB, ABC.

1. The segments GG_i are the homothetic transforms $(G_i', \frac{1}{4})$ of the segments AA_1, BB_1, CC_1, DD_1, hence the lines GG_i are normal to the opposite faces BCD, CDA, DAB, ABC. The same statement holds for the segments $O_{12}O_{12i}$, the homothetic transforms $(O, 3/2)$ of GG_i. Furthermore, since the distances of the centers O_{12} and O_{12i} of the twelve point spheres (O_{12}) and (O_{12i}) of T and T_i from the planes of faces BCD, CDA, DAB, ABC are equal† to

$$AA_1/6, \qquad BB_1/6, \qquad CC_1/6, \qquad DD_1/6,$$

$O_{12}$and O_{12i} are symmetric with respect to the planes of the faces of T. But the spheres (O_{12}), (O_{12i}) have the same radius $R/3$ (where R is the circumradius of T); hence these spheres are symmetric with respect to the faces of T. From this follows the following result.

THEOREM. *In a tetrahedron $ABCD$ for which the altitudes AA', BB', CC', DD', cut the circumsphere in A_1, B_1, C_1, D_1, the tetrahedra A_2BCD, B_2CDA, C_2DAB, D_2ABC, where A_2, B_2, C_2, D_2 are symmetric to A_1, B_1, C_1, D_1 with respect to the planes of the opposite faces of $ABCD$, have the same twelve point sphere as $ABCD$.*

2. In a triangle ABC whose altitudes AA', BB', CC', cut the circumcircle in points A_1, B_1, C_1 (other than the vertices), the nine point circles of triangles A_1BC, B_1CA, C_1AB are symmetric to the nine point circle (O_9) of ABC with respect to the sides BC, CA, AB. From this we have the classical theorem which follows.

THEOREM. *In a triangle ABC, of orthocenter H, the triangles HBC, HCA, HAB, ABC, have the same nine point circle.*

3. This establishes a complete analogy between the triangle and the tetrahedron.

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* N. A. Court, Modern Pure Solid Geometry, p. 250.

† V. Thébault, Bulletin de l'Ecole Polytechnique de Timisoara (Roumania), 1944, p. 25; R. Blanchard, Mathesis, t. 55, p. 299.

NOTE ON A PAPER BY L. S. JOHNSTON

IVAN NIVEN, University of Oregon

In a paper entitled *Denumerability of the Rational Number System* in the February, 1948, issue of this MONTHLY, L. S. Johnston has exhibited a one-to-one correspondence between the positive integers and the rational numbers, with a specific construction provided. The purpose of this note is to give another constructive correspondence.

For any positive integer n, let r be the number of zeros occurring in the representation of n in the binary scale of notation. These r zeros divide the representation into $r+1$ blocks of 1's, some blocks being perhaps empty: for $j=2, 3, \cdots, r$ let a_j denote the number of 1's between the $(j-1)$th and the jth zeros, counted from the right; define a_1 and a_{r+1} as the number of consecutive 1's on the right and left ends of the representation. E.g., if n is 110010 in the binary scale, then $r=3$ and $(a_1, a_2, a_3, a_4)=(0, 1, 0, 2)$. If $n=1$, then $r=0$, and $a_1=1$. Note that n determines a unique set of values $a_1, a_2, \cdots, a_{r+1}$, with positive a_{r+1}, and conversely.

Now define $b_j=-a_j/2$ for even a_j, and $b_j=(a_j+1)/2$ otherwise: this is a one-to-one correspondence between the non-negative integers a_j and all integers b_j. Finally, if $p_1, p_2, p_3, \cdots$ are the consecutive primes 2, 3, 5, $\cdots$, we define

$$f(n)=\prod_{j=1}^{r+1} p_j^{b_j}. \tag{1}$$

Thus to any positive integer n corresponds a unique positive rational number $f(n)$. Conversely, any positive rational number f, except 1, can be written uniquely in the form (1) with $b_{r+1}\neq 0$. The exponents b_j determine a unique set a_j, with $a_{r+1}\neq 0$, and these give a unique positive integer n. E.g., if $f=33/49$, then $(b_1, b_2, b_3, b_4, b_5)=(0, 1, 0, -2, 1)$ and $(a_1, a_2, a_3, a_4, a_5)=(0, 1, 0, 4, 1)$, so that $n=1011110010$ in the binary scale.

We now obtain a one-to-one correspondence between the positive integers and all positive rational numbers, by letting $n+1$ correspond to $f(n)$, written $n+1\leftrightarrow f(n)$, for all positive integers n, and in addition we let $1\leftrightarrow 1$. On the other hand, if we want a one-to-one correspondence between the positive integers and all rational numbers, we can use the correspondences $1\leftrightarrow 0$, $2\leftrightarrow 1$, $3\leftrightarrow -1$, $2n+2\leftrightarrow f(n)$, $2n+3\leftrightarrow -f(n)$, where n ranges over all positive integers.

A GENERALIZATION OF EULER'S ϕ-FUNCTION

V. L. KLEE, JR., University of Virginia

The purpose of this note is to point out an interesting generalization of Euler's ϕ-function, and to show that to a large extent its properties parallel those of the ϕ-function. Proofs are omitted, but may be supplied readily by the reader. It is suggested that many of the theorems stated below would be suitable as exercises for students in a Number Theory class.

We call an integer kth-*power-free* if it is not divisible by the kth power of any integer >1. If k and n are positive integers we denote by $\Phi_k(n)$ the number of integers h in the set $1, \cdots, n$, for which the greatest common divisor (h, n) is kth-power-free. Φ_1 is the ϕ-function.

In establishing for Φ_k properties analogous to those of ϕ one can follow the order and method of proof of Landau [Elementare Zahlentheorie] to obtain the following results:

$$\sum_{d|n} \Phi_k(d^k) = n^k;$$

$$\Phi_k(n^k) = n^k \sum_{d|n} \mu(d)/d^k;$$

$$\Phi_k(n) = n \prod_{p \text{ prime and } p^k|n} (1 - p^{-k});$$

$$\Phi_k(\prod p_i^{a_i}) = \prod_{a_i<k} p_i^{a_i} \cdot \prod_{a_i \geqq k} p_i^{a_i-k}(p_i^k - 1),$$

for distinct primes p_i; Φ_k is multiplicative.

From these results one obtains the following:

$\Phi_k(mn)/[\Phi_k(m)\Phi_k(n)] = Q\Phi_k(P)/P\Phi_k(Q)$, *where* P *is the product of the* k*th powers of all primes* p *for which* $p^k|m$ *and* $p^k \nmid n$ *but* $p^k|mn$, *and* Q *is the product of the* k*th powers of all primes* q *for which* $q^k|m$ *and* $q^k|n$.

$\Phi_k(n)$ *is odd if and only if* $n = 2^e m$, *where* m *is odd and* k*th-power-free and either* $e = 0$ *or* $e = k$.

The analogue of the Fermat theorem which holds for Φ_k *is that if* a *and* n *are coprime, then* $n/(a^{\Phi_k(n)} - 1, n)$ *is* k*th-power-free.*

The reader will be able to supply other results of similar nature.

As a generalization, different from that above, of the relation $\phi(n) = n\sum_{d|n}\mu(d)/d$, we have $\Phi_k(n) = n\sum_{d|n}\mu^k(d)/d$, where $\mu^k(n)$ is defined as follows: $\mu^k(1) = 1$; if $n = p_1^{a_1}p_2^{a_2} \cdots p_t^{a_t}$ is the canonical factorization of n, then $\mu^k(n) = (-1)^t$ if $a_1 = a_2 = \cdots = a_t = k$, and $\mu^k(n) = 0$ if for some i, $a_i \neq k$.

It may be of interest to note the following properties of the functions μ^k: μ^k *is multiplicative;* $\mu^1(n^1) = \mu^2(n^2) = \cdots$; $\sum_{d|n^k}\mu^k(d)/d^j = \sum_{d|n^j}\mu^j(d)/d^k$; $\sum_{d|n}\mu^k(d)$ *is* 1 *or* 0 *according as* n *is or is not* k*th-power-free.*

We note in conclusion a further possible generalization. If A is an arbitrary set of integers and k and n are positive integers, denote by $\Phi_{(A,k)}(n)$ the number of residue classes R modulo n for which $r \in R$ and $a \in A$ implies that $(r+a, n)$ is kth-power-free. $\Phi_{(A,1)}$ is Lucas's generalization [*Theorie des Nombres*, p. 402]. By a simple adaptation of the argument of Dickson [*Introduction to the Theory of Numbers*, p. 7] the following result can be established.

$\Phi_{(A,k)}$ *is multiplicative. If* $x = \prod p_i^{a_i}$ *is the canonical factorization of* x, *then* $\Phi_{(A,k)}(x) = \prod(p_i^{a_i} - r_i p_i^{a_i-k})$, *where* $r_i = 0$ *for* $a_i < k$, *and for* $a_i \geqq k$, r_i *is the number of distinct residues modulo* p_i^k *of the integers in* A.

CORRECTIONS

V. L. Klee, *Some remarks on Euler's totient*, this MONTHLY, vol. 54, 1947, p. 332. The last line should be replaced by:

$n+2$ are prime, n is twice a Mersenne prime, or n has the form $4p$, where p and $2p+1$ are prime.

R. W. Hamming, *Subseries of monotone divergent series*, this MONTHLY, vol. 54, 1947, pp. 462–463. The theorem which was proved in the note should be stated correctly as follows:

THEOREM. *In order that an increasing sequence $I(n)$ of positive integers may be such that $\sum d_{I(n)}$ is divergent whenever $\sum d_n$ is a divergent series of positive terms for which $d_1 \geqq d_2 \geqq \cdots$, it is necessary and sufficient that $I(n)/n$ be a bounded sequence.*

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

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EQUATIONS AND LOCI IN POLAR COÖRDINATES

R. W. WAGNER, Oberlin College

The purpose of this note is to further publicize and to urge a wider application of an idea discussed by C. Fox* to clarify the difficulties which are encountered in seeking the intersections of loci defined by equations in polar coördinates.

Everyone who has worked with polar coördinates is aware of the fact that a point has an infinite number of coördinates. Thus, the point with coördinates (r, θ) has the other coördinates $([-1]^n r, \theta+n\pi)$, $n=0, \pm 1, \pm 2, \cdots$. Writing the coördinates in this way implies that the common conventions that r may be any real number and that θ is measured in radians are being used.

The locus or graph of an equation is usually taken to mean the set of all points for which some pair of coördinates satisfy the equation.

Fox's idea is this: *The lack of uniqueness in the coördinates of a point leads, in general, to a lack of uniqueness in the equation of a locus.* This principle is overlooked or ignored in the majority of text books which the author has examined. A few books† mention it and use it to find the points of intersection of polar loci.

* C. Fox, The Polar Equations of a Curve, Math. Gazette, vol. 15 (1931), pp. 486–7.

† They are the analytic geometry texts by Curtiss and Moulton (1930), by Middlemiss (1947), by Nathan and Helmer (1947), by Randolph and Kac (1946), by Sisam (1926), by Nowlan (1932), and by Mason and Hazzard (1935).

However, none of these books use the principle to its full extent, for example, to clarify the tests for symmetry or to expedite the plotting of curves.

To investigate the validity of this principle, consider the locus defined by the equation

$$F(r, \theta) = 0. \tag{1}$$

A point will be on this locus if, and only if, for some integer n $(n=0, \pm 1, \pm 2, \cdots)$

$$F([-1]^n r, \theta + n\pi) = 0. \tag{2}$$

The locus defined by assigning any permissible value to n in equation (2) will be the same as the locus defined by equation (1). For, any point which has a pair of coördinates which satisfy (1) will have a pair which satisfy (2), and conversely.

The equations obtained from (2) by assigning various values to n may be algebraically equivalent. This cannot occur unless $F(r, \theta)$ is a periodic function of θ with a period which is a rational multiple of π. If this period be $p\pi/q$, each equation obtained from (2) is algebraically equivalent to one of at most $2p$ equations which are algebraically non-equivalent. For example, all alternative forms for the equation

$$r - a \cos \theta - b \sin \theta = 0$$

are algebraically equivalent because

$$(-1)^n r - a \cos (\theta + n\pi) - b \sin (\theta + n\pi) = \begin{cases} r - a \cos \theta - b \sin \theta & (n \text{ even}) \\ -r + a \cos \theta + b \sin \theta & (n \text{ odd}). \end{cases}$$

For another example, there are four algebraically non-equivalent forms for the equation of the locus defined by

$$r - \cos (\theta/2) = 0. \tag{3}$$

The other forms are

$$-r + \sin (\theta/2) = 0, \quad r + \cos (\theta/2) = 0, \quad \text{and} \quad -r - \sin (\theta/2) = 0.$$

The student should be made aware of the lack of uniqueness of the polar equation of a locus from the start. The first step in the discussion of the locus of an equation should be to find the alternate forms of the equation. The students are delighted to learn that they can graph several equations at once!

Plotting. By using the alternate equations one can frequently shorten the table values or the work required to prepare it by listing a value of r for each of the alternate equations and restricting θ to the interval $0 \leqq \theta \leqq \pi$. For example, to plot the graph of equation (3), the table of values would have values of θ in one column and values of r in four parallel columns. Each of these latter columns would show the values of r obtained from one of the alternate equations.

Symmetry. The usual tests for symmetry of a polar locus are based on condi-

tions which are sufficient to show symmetry. The failure of the test to show symmetry does not show the lack of symmetry. By utilizing the alternate equations of the locus one can state a necessary and sufficient test for symmetry which requires a single substitution for each type of symmetry.

The locus of $F(r, \theta)$ *is symmetrical with respect to* (i) *the pole,* (ii) *the polar axis or* (iii) *the perpendicular to polar axis at the pole if, and only if,* (i) $F(-r, \theta)=0$, (ii) $F(r, -\theta)=0$, *or* (iii) $F(-r, -\theta)=0$ *is algebraically equivalent to* $F(r, \theta)=0$ *or one of its alternate forms.*

After a type of symmetry has been found, some of the r-columns of the table of values may be omitted and the symmetry used to complete the curve.

Intersections. The points of intersection are found by solving simultaneously all the pairs of equations made up of an equation for the first locus and an equation for the second locus. This can be systematically accomplished by first introducing an integer valued parameter into each equation, by then restricting the values of the parameters to sets of values which yield algebraically non-equivalent equations, and finally solving the two equations simultaneously in terms of the two parameters. If there are but few alternate forms for each equation, it may be simpler to solve the various pairs of equations than to solve the equations involving the parameters.

This procedure does not include the pole among the points of intersection unless the two curves have a common tangent at the pole. In general, an intersection at the pole is most readily found by noting that both curves pass through the pole.

Example. Find the intersections of the cardioid and the limaçon whose equations are

$$r = 2 + 2\cos\theta, \qquad r = 10\cos\theta - 2.$$

Each of these equations has an alternate form, namely,

$$r = 2\cos\theta - 2, \qquad r = 10\cos\theta + 2.$$

If one solves the two given equations simultaneously, he finds the points on the inner loop of the limaçon $(3, \pm\pi/3)$. The simultaneous solutions of the two alternate equations are the same points but the coördinates appear in the form $(-3, \pm 2\pi/3)$. The simultaneous solutions of a pair of equations made up of one given equation and the alternate of the other equation are the points on the outer loop of the limaçon $(2, \pm\pi/2)$, which may appear in the form $(-2, \pm\pi/2)$. The pole is also an intersection.

Conics. If the equation of a conic

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \tag{4}$$

is changed to polar coördinates, the polar coördinate equation is factorable. If the conic passes through the pole, one factor leads to the equation $r=0$ and the other factor leads to the unique equation of the conic. If the conic does not pass through the pole, it has two equations and these equations are obtained by

equating the factors to zero individually.

The result of substituting $x=r\cos\theta$ and $y=r\sin\theta$ into equation (4) is

$$(5)\quad (A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta)r^2 + 2(D\cos\theta + E\sin\theta)r + F = 0,$$

which can be abbreviated to

$$(6)\qquad Hr^2 + 2Gr + F = 0.$$

When θ is increased by π, H is not changed and the sign of G is changed. If the conic passes through the pole, $F=0$ and equation (6) may be factored into

$$r = 0 \quad \text{and} \quad Hr + 2G = 0.$$

This second equation has no alternate forms because the change which is made to find an alternate equation merely changes the sign of both terms. If the conic does not pass through the pole, $F\neq 0$ and, after multiplication by H, (6) may be factored into

$$Hr + G + \sqrt{G^2 - HF} = 0 \quad \text{and} \quad Hr + G - \sqrt{G^2 - HF} = 0.$$

Each of these equations is an alternate form of the other because the substitution which is used to find alternate forms changes each one into the other. Also they are the only algebraically non-equivalent forms of the equation.

This illustrates a situation which should be true in general. If the rectangular equation of a locus be changed to polar coördinates, the polar equation is factorable and the factors are a complete set of alternate equations for the locus.

MAXIMUM UNCERTAINTY AS A SIMPLE EXAMPLE OF A NON-DISTRIBUTIVE ALGEBRA

L. C. Green, Haverford College

Practical examples of non-distributive algebras simple enough for classroom use are difficult to find. It is the purpose of this note to point out that the computation of the maximum uncertainty in the elementary theory of errors provides such an example. By the elementary theory is meant that part of the theory of errors in which the *absolute* uncertainty of the sum or difference of two quantities is taken as the sum of their *absolute* uncertainties and the *percentage* uncertainty in the product or quotient of two quantities is taken as the sum of their *percentage* uncertainties. So long as the same measure of uncertainty is used throughout a discussion, it makes no difference in the application of these rules which one of the various measures is used, that is, average deviation, standard deviation and so forth.

Consider the illustration of determining the maximum uncertainty in the expansion of a rod of soda glass $100.00\pm.01$ cm. in length if the temperature increases from $20.0\pm.1$°C to $22.5\pm.1$°C and if the coefficient of linear expansion for soda glass is $(1.2\pm.1)\cdot 10^{-5}$ per degree C. from the formula

$$\Delta L = Lk(T_2 - T_1).$$

If we evaluate the parenthesis first we obtain the result $(3.0 \pm .5) \cdot 10^{-3}$ cm. but if we make use of the distributive law we find $(3.0 \pm 4.5) \cdot 10^{-3}$ cm. By the latter method the uncertainty in the expansion would appear to be 50% greater than the expansion itself. Clearly the first result is the one of interest.

In order to make the situation explicit let us define a number pair (x, u) where x represents some measured quantity and u is its absolute uncertainty. The fundamental operations then take the form

$$(x_1, u_1) \pm (x_2, u_2) = (x_1 \pm x_2, u_1 + u_2)$$
$$(x_1, u_1) \cdot (x_2, u_2) = (x_1 x_2, u_1 x_2 + u_2 x_1).$$

It is evident that addition and multiplication are both associative and commutative. On the other hand we have

$$(x_1, u_1)[(x_2, u_2) \pm (x_3, u_3)] = (x_1[x_2 \pm x_3], u_1[x_2 \pm x_3] + [u_2 + u_3]x_1).$$

In contrast the usual distributive law would yield

$$(x_1, u_1)[(x_2, u_2) \pm (x_3, u_3)] = (x_1[x_2 \pm x_3], u_1[x_2 + x_3] + [u_2 + u_3]x_1).$$

This non-distributive character has its origin in the fact that with the content assigned above to the symbols, $-(x, u) = (-x, u)$. If we add the rule of reckoning that $-(x, u)$ is to be replaced by $(-x, u)$ whenever it appears, the algebra becomes distributive.

Finally for their curiosity, one may write the relations:

$$\frac{(x_1, u_1)}{(x_2, u_2)} = \left(\frac{x_1}{x_2}, \frac{u_1 x_2 + u_2 x_1}{x^2}\right),$$
$$(x, u)^n = (x^n, nux^{n-1}),$$
$$(x, u)^{1/n} = \left(x^{1/n}, \frac{1}{n} ux^{(1/n)-1}\right).$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 821. *Proposed by B. H. Brown, Dartmouth College*

A sports promoter hired nine not too scrupulous athletes, and formed three

cross-country teams A, B, and C, of three men each, which he took on tour for a series of dual and triple meets. If a team could win, it always would; but a losing team could be bribed to run more slowly. Except for this failing, the athletes always ran as automata, and always finished in the same order, with no ties. (In a cross-country meet the winner receives 1 point, the next man 2 points, *etc.* The team score is the sum of the points received by its members, and low score wins.)

In dual meets, team A beat team B, B beat C, and C beat A, much to the annoyance of the promoter who unjustly accused the men of dishonesty.

In an honest triple meet, team A won, whereupon team C went to team B with the following dishonest but logical offer: "No conceivable dishonesty on the part of our C team can enable your B team to beat or even tie A; but if you will slow down, our C team can beat A, and we will divide the profits with you."

Determine completely the composition of each team in terms of the relative excellence of its members.

E 822. *Proposed by W. R. Ransom, Tufts College*

Every factorial that can be expressed as the difference of two squares can be so expressed in two different ways.

E 823. *Proposed by Max LeLeiko, Rutgers University*

Find four distinct non-zero integers a, b, c, d such that $a^2+b^2+c^2+d^2$

$$= (1/5)[(a+b)^2+(a+c)^2+(a+d)^2+(b+c)^2+(b+d)^2+(c+d)^2]$$
$$= (1/7)[(a+b+c)^2+(a+b+d)^2+(a+c+d)^2+(b+c+d)^2]$$
$$= (1/3)(a+b+c+d)^2.$$

E 824. *Proposed by E. P. Starke, Rutgers University*

We modify the harmonic series by taking the first term positive, the next two negative, the next three positive, *etc.* Show that this modified series is convergent.

E 825. *Proposed by W. E. Patten, Spartanburg, S. C.*

If p points and no more are collinear, we say they form a (collinear) set of *multiplicity* p.

Suppose we are given the rectangular lattice of points which are the corners of the squares of an h by v checkerboard, where $h \leqq v$. Determine the number of collinear sets, for each possible multiplicity $\geqq 2$, occurring in the set of points.

SOLUTIONS

An Impossible Journey

E 788 [1947, 471]. *Proposed by Leo Moser, University of Manitoba*

Consider a map on a spherical surface where the countries are determined by n great circles of which no three are concurrent. Show that if n is a multiple of

four it is impossible to make a trip visiting each country once and only once, if travelling along a boundary or crossing at a boundary point of more than two countries is forbidden.

Solution by the Proposer. We first prove by induction that such a map may be colored with two colors, so that no two countries with common boundary have the same color. This is certainly true for $n=1$. Suppose it is true for $n-1$. When we add another great circle, it is only necessary to reverse all the colors to one side of this circle to obtain a coloring for n. Also, if we use this coloring, it is clear that for n even, diametrically opposite countries will have the same color. Hence, in this case, there will be an even number of countries of each color. We next show that these two even numbers cannot be equal. If they were equal, then the total number of countries, F, would be a multiple of four, whereas we shall show that $F=4k+2$.

Since each circle intersects each other circle twice, the number of vertices, V, is $n(n-1)$. Now every vertex belongs to four edges, and every edge has two vertices. Therefore the number of edges, E, is $2V$. Using Euler's theorem, $V-E+F=2$, we get $F=n(n-1)+2$, and if n is a multiple of four, F will be of the form $4k+2$.

Finally, we note that in any trip on a map colored by our method, one must change color with every crossing from country to country, and since the number of countries of one color differs from the number of countries of the other color by at least two it will be impossible to reach them all.

Note. In a way this is a generalization of E 711 [1946, 593], because the reciprocal map for $n=4$ of our problem is homeomorphic to the surface of the rhombic dodecahedron. The direct generalization of E 711, which is also true, states that there is no Hamilton path for any polyhedron all of whose faces are parallelograms, if the number of faces is $n(n-1)$, (see problem 4176 [1947, 169]), where n is a multiple of four.

Derivatives of tan x

E 789 [1947, 471]. *Proposed by Kaidy Tan, Chip-Bee Institute, Amoy, Fukien, China*

If $y=\tan x$, show that

$$(d^n y/dx^n)\cos^{n+1} x = \begin{vmatrix} \cos x & 0 & \cdots & \sin x \\ \cos(x+\pi/2) & \cos x & \cdots & \sin(x+\pi/2) \\ \cos(x+2\pi/2) & 2\cos(x+\pi/2) & \cdots & \sin(x+2\pi/2) \\ \cdots & \cdots & \cdots & \cdots \\ \cos(x+n\pi/2) & n\cos(x+\overline{n-1}\,\pi/2) & \cdots & \sin(x+n\pi/2) \end{vmatrix}.$$

Solution by N. J. Fine, University of Pennsylvania. Let $w=\sin x$, $z=\cos x$, so that $w=yz$. If we differentiate r times by the Leibnitz rule, we have

$$w^{(r)} = \sum_{k=0}^{r} \binom{r}{k} y^{(k)} z^{(r-k)} \qquad (r = 0, 1, 2, \cdots, n).$$

Regard this as a system of $n+1$ linear equations in the quantities $y^{(k)}$, $k=0, 1, 2, \cdots, n$. The determinant of the system, being triangular and having $z^{(0)}=\cos x$ down the main diagonal, has the value $\cos^{n+1}x$. Hence, solving for $y^{(n)}$, we find

$$y^{(n)} \cos^{n+1} x = \begin{vmatrix} z^{(0)} & . & . & . & \cdots & w^{(0)} \\ z^{(1)} & z^{(0)} & . & . & \cdots & w^{(1)} \\ z^{(2)} & \binom{2}{1} z^{(1)} & z^{(0)} & . & \cdots & w^{(2)} \\ z^{(3)} & \binom{3}{1} z^{(2)} & \binom{3}{2} z^{(1)} & z^{(0)} & \cdots & w^{(3)} \\ . & . & . & . & \cdots & . \\ z^{(n)} & \binom{n}{1} z^{(n-1)} & . & . & \cdots & w^{(n)} \end{vmatrix}$$

We now observe that $z^{(k)}=\cos(x+k\pi/2)$, $w^{(k)}=\sin(x+k\pi/2)$, and the proof is complete.

Also solved by the proposer.

Editorial Note. In connection with this problem see *Successive derivatives of tan x*, by R. K. Moreley, National Mathematics Magazine, Vol. XIX, No. 6.

Algorithm for Approximating $a^{1/n}$

E 790 [1947, 471]. *Proposed by H. S. Wall, University of Texas*

Let

$$f(x, y) = \frac{a + xy(x^{n-2} + x^{n-3}y + x^{n-4}y^2 + \cdots + y^{n-2})}{x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}},$$

where n is an integer greater than unity, and $a>0$. If $x_0>0$, $x_1>0$, and x_2, x_3, x_4, $\cdots$ are computed recurrently by means of the formula

$$x_{p+2} = f(x_p, x_{p+1}), \qquad p = 0, 1, 2, \cdots$$

then

$$\lim_{p=\infty} x_p = a^{1/n}.$$

I. *Solution by the Proposer.* We may write

$$f(x, y) = f(y, x) = x - \frac{x^n - a}{x^{n-1}\left(1 + \dfrac{y}{x} + \dfrac{y^2}{x^2} + \cdots + \dfrac{y^{n-1}}{x^{n-1}}\right)}. \tag{1}$$

Therefore $f(a^{1/n}, y) = f(x, a^{1/n}) = a^{1/n}$, and the statement is obvious if x_0 or x_1 equals $a^{1/n}$.

Formula (1) shows that $f(x, y)$ is an increasing function of y, for fixed x, if $x^n > a$, and a decreasing function of y, for fixed x, if $x^n < a$. Since $f(x, y) = f(y, x)$, we may interchange x and y in this statement.

If $x_0, x_1 > a^{1/n}$, then $x_2 = f(x_0, x_1) > a^{1/n}$. For, if we let $x_0 \to a^{1/n}$, x_2 must *decrease* to $a^{1/n}$. Moreover, $x_2 < x_0, x_1$ inasmuch as

$$x_2 - y = -\frac{y^n - a}{y^{n-1}\left(1 + \frac{x}{y} + \frac{x^2}{y^2} + \cdots + \frac{x^{n-1}}{y^{n-1}}\right)} < 0,$$

for $y = x_0, x_1$. It follows that $x_2 > x_3 > x_4 > \cdots > a^{1/n}$, and therefore $\lim_{p=\infty} x_p = r \geqq a^{1/n}$. Since $r = f(r, r) = [a + (n-1)r^n]/nr^{n-1}$, then $r^n = a$, $r = a^{1/n}$.

If $x_0 < a^{1/n} < x_1$, then $x_2 = f(x_0, x_1)$ is a decreasing function of x_1. If we let $x_1 \to a^{1/n}$, then x_2 must *increase* to $a^{1/n}$, so that $x_2 < a^{1/n}$. Moreover, $x_0 < x_2$. For, if $x_2 < x_0$, then x_2 must increase through the value x_0 as $x_1 \to a^{1/n}$, which is impossible inasmuch as $x_0 < a^{1/n}$. In this case we must have $x_2 < x_3 < a^{1/n}$. If $x_1 < a^{1/n} < x_0$, then $x_1 < x_2 < a^{1/n}$. Hence we have only to consider the case where $x_0 < x_1 < a^{1/n}$. But then, $x_2 > a^{1/n}$, $x_1 < x_3 < a^{1/n}$, $x_3 < x_4 < a^{1/n}$, $x_5 > a^{1/n}$, $x_4 < x_6 < a^{1/n}$, $x_6 < x_7 < a^{1/n}$, $x_8 > a^{1/n}$, $\cdots$. Therefore, the sequence $x_1, x_3, x_4, x_6, x_7, x_9, x_{10}, \cdots$ converges to $a^{1/n}$, and $x_{3n+2} = f(x_{3n}, x_{3n+1}) \to f(a^{1/n}, a^{1/n}) = a^{1/n}$.

II. *Solution by Leo Moser, University of Manitoba.* The result is made intuitive by the following considerations:

(a) If $x_p = x_{p+1}$, then $f(x_p, x_{p+1})$ is the approximation obtained for $a^{1/n}$ by using Newton's method on the approximation x_p.

(b) If $x_p \neq x_{p+1}$, then $f(x_p, x_{p+1})$ is the x-coördinate of the point of intersection of the x-axis and the straight line joining the points on the curve $y = x^n - a$, having x-coördinates x_p and x_{p+1}.

This would indicate the possible lines of generalization of the given result. Thus we could obtain similar methods for solving a much larger class of equations than $y^n = a$ (n an integer). Also, we might work with more than two points on the curve at a time and pass special curves through these, say circles through three points.

Also solved by J. G. Herriot.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4300. *Proposed by Leo Moser, University of Toronto*

Let $a_1, a_2, \cdots, a_n$ be n, not necessarily distinct, elements of a group of order n. Show that there exist integers p and q, $1 \leqq p \leqq q \leqq n$, such that

$$\prod_{i=p}^{q} a_i = 1.$$

4301. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider similar triangles ABC, the sides BC, CA, AB of which pass through the fixed points A_1, B_1, C_1. (1) The locus of the circumcenters O of these triangles is a circle. (2) The circumcircles (O) are orthogonal to a fixed circle. (3) When the points A_1, B_1, C_1 are collinear, the envelope of the circles (O) is a cardioid. What is this envelope otherwise?

4302. *Proposed by Joseph Rosenbaum, the Milford School, Conn.*

Prove that if x and y have no common factor then every odd factor of

$$x^{2^n} + y^{2^n},$$

where n is a positive integer, is of the form $2^{n+1}m+1$.

4303. *Proposed by G. T. Williams, Cambridge, Mass.*

If

$$\theta_n = \int_0^1 x(x-1)\cdots(x-n+1)dx,$$

and

$$\phi_n = \int_0^1 x(x+1)\cdots(x+n-1)dx,$$

show that $(-1)^n\theta_n = \phi_n - n\phi_{n-1}$ and find ϕ_n in terms of θ_n.

4304. *Proposed by H. D. Grossman, New York City*

From the vertices of a regular n-gon three are chosen to be the vertices of a

triangle. Prove that the number of essentially different possible scalene triangles is the integer nearest to $(n-3)^2/12$. Compare 3893 [1940, 664].

SOLUTIONS

Squares of Special Form

4237 [1947, 112]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In which systems of numeration, of base B, are there two-digit squares $m^2=ab$ and $n^2=cd$, if $d=a+1$ and $b=c+1$? Show that, if also $b+d=B+1$, the four-digit numbers $ddbb$ and $bbdd$ are the squares of the numbers mm and nn.

Solution of E. P. Starke, Rutgers University. From the first hypothesis we have

$$m^2 = aB + b, \qquad n^2 = cB + d, \tag{1}$$

with $d=a+1$ and $b=c+1$. It is evidently necessary that b and d be quadratic residues mod B of a and c, respectively. Upon eliminating B we have $c(m^2-b)=a(n^2-d)$ or

$$cm^2 - an^2 = (c - a)(c + a + 1). \tag{2}$$

Hence, if $(c, a)=1$, any solution m, n, of (2) will satisfy (1) and determine integral values of B. The values of m and n must be sufficiently large to provide that $B>m, n$. For example, with $a=7$, $c=1$, (2) becomes $m^2-7n^2=-54$ which leads to an infinity of solutions, $n=5$, $m=11$, $B=17$; $n=7$, $m=17$, $B=41$; and so on.

Under the additional hypothesis we have at once $a+b=B$ and $m^2=aB+b$, whence

$$\begin{aligned}(mB + m)^2 &= m^2B^2 + 2m^2B + m^2 = aB^3 + (b + 2a)B^2 + (2b + a)B + b \\ &= (a + 1)B^3 + (a + 1)B^2 + bB + b = dB^3 + dB^2 + bB + b,\end{aligned}$$

as required. Upon interchange of b and d, the corresponding equation for nn^2 results. To find numerical examples we put (1) in the form

$$m^2 - 1 = d(B - 1), \qquad n^2 - 1 = b(B - 1), \tag{3}$$

$$m^2 + n^2 = (b + d)(B - 1) + 2 = B^2 + 1. \tag{4}$$

If we put $m+1=B-1$, $m-1=d$, we have

$$b = B + 1 - d = 4, \qquad n^2 = 1 + b(B - 1) = 4B - 3.$$

Since n is odd we put $n^2=(2p+1)^2$, whence

$$B = p^2 + p + 1$$

and we have solutions for all $p>1$.

We may also put $x(m-1)=y(B-1)$, $y(m+1)=dx$, with any choice of x and y. In particular if $x=2$, $y=1$, we have $B=4d-3$, $b=3d-2$, $n^2=12d^2-20d+9$

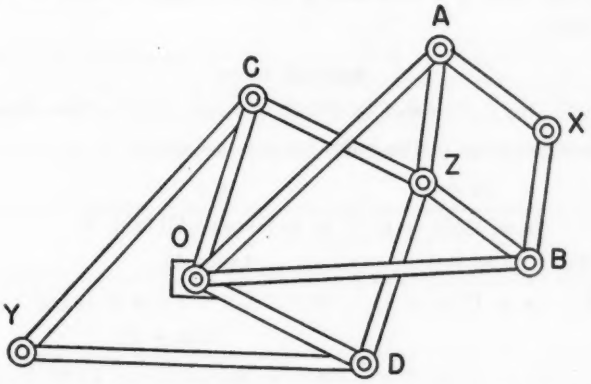
or $(6d-5)^2-3n^2=-2$ which leads to solutions such as $d=45$, $n=153$, $B=177$, $m=89$. $x=3$, $y=2$ leads to solutions like $d=140$, $n=233$, $m=209$, $B=313$. $x=4$, $y=3$ gives $d=150$, $n=175$, $m=199$, $B=265$ and other solutions.

No general formula giving all solutions appears likely. To test whether a proposed value of B is possible, we have two strong necessary conditions: (a) B^2+1 must be divisible by two or more distinct odd primes in order to be a sum of two squares in more than one way, as in (4); and (b) $B-1$ must be divisible by 4 or by two distinct odd primes, otherwise in (3) m^2, $n^2\equiv 1 \pmod{B-1}$ have only the solutions m, $n=1$, $B-2$, which will not satisfy (4).

Peaucellier Cells

4239 [1947, 168]. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

$AXBZ$ is a jointed rhombus connected with a fixed point O by two equal rods OA, OB. $OCZD$ is a jointed rhombus and YC, YD are equal rods. (Two Peaucellier cells, as it were "cross joined.") Prove that as Y describes a circle, X describes a conic.



Solution by the Proposer. Using the well known property of Peaucellier cells*

$$OZ\cdot OX = OA^2 - ZA^2 = m^2,$$
$$YZ\cdot YO = YC^2 - ZC^2 = n^2.$$

Therefore the following relations hold:

$$OY(OY - OZ) = OY^2 - OY\cdot OZ = n^2,$$
$$OX = \frac{OY\cdot m^2}{OY^2 - n^2}.$$

* See J. D. C. De Roos, Linkages: The Different Forms and Uses of Articulated Links, Van Nostrand, 1879, pp., 87ff. See also R. C. Yates, Tools, Baton Rouge 1941, p. 184, Fig. 1; and also Yates, Curves, West Point 1946, pp. 29, 52–54.

That is, in the notation of complex variables,

$$X = \frac{Ym^2}{Y\overline{Y} - n^2},$$

or

$$Y = \theta X, \qquad \overline{Y} = \theta \overline{X}, \qquad \theta^2 X \overline{X} - n^2 = \theta m^2,$$

where

$$\theta = (Y\overline{Y} - n^2)/m^2,$$

a real quantity. Hence, if Y moves upon the circle

$$\alpha Y\overline{Y} + c\overline{Y} + \bar{c}Y + \beta = 0,$$

then X traces the conic

$$(\alpha n^2 + \beta)^2 X \overline{X} + \{n^2(c\overline{X} + \bar{c}X) - m^2\beta)\}\{c\overline{X} + \bar{c}X + m^2\alpha\} = 0.$$

Note further that if Y describes a circle through O, then the point Z describes a cissoid.

Telescopic Series

4242 [1947, 168]. *Proposed by W. O. Pennell, Exeter, New Hampshire*

Determine the sums of the following infinite series:

$$(1) \qquad \frac{n+2}{n+1} - \frac{2n+2}{(n+1)(2n+1)} + \frac{3n+2}{(n+1)(2n+1)(3n+1)} - \cdots,$$

$$(2) \qquad \frac{3n+7}{n+1} - \frac{8n+9}{(n+1)(2n+1)} + \frac{15n+11}{(n+1)(2n+1)(3n+1)} - \frac{24n+13}{(n+1)(2n+1)(3n+1)(4n+1)} - \cdots,$$

where n is any real number except $0, -1, -1/2, -1/3$, and so on.

I. *Solution by P. A. Clement, University of California at Los Angeles.* Consider a series of the type

$$(3) \qquad S = \sum_{k=1}^{\infty} (-1)^{k+1} \left[\frac{a_k}{(n+1)(2n+1)\cdots(\overline{k-1}\cdot n+1)} + \frac{b_k}{(n+1)(2n+1)\cdots(kn+1)} \right]$$

satisfying

$$(a) \qquad a_{k+1} = b_k,$$

(b) $$\lim_{k\to\infty} \frac{b_k}{\prod_{s=1}^{k}(sn+1)} = 0,$$

(c) $n \neq -1, -1/2, -1/3, \cdots$; and $n \neq 0$ when $\lim b_k \neq 0$.

It is easy to see that the series telescopes so that the sum of the first k terms is given by

$$S_k = a_1 + (-1)^{k+1} \frac{b_k}{\prod_{s=1}^{k}(sn+1)},$$

so that, by (b) and (c), $S=a_1$.

The general term of (3) by addition becomes

$$(-1)^{k+1} \frac{a_k kn + a_k + b_k}{\prod_{s=1}^{k}(sn+1)}.$$

This is easily verified to be the general term of (1) if $a_k=b_k=1$, and of (2) if $a_k=k+2$ and $b_k=k+3$, and these specifications obviously satisfy (b) above. Hence, for the series (1), $S=a_1=1$, and for (2), $S=a_1=3$.

II. *Solution by C. F. Pinzka, Student, Rutgers University.* Let

$$S_1 = 1 - \frac{1}{n+1} + \frac{1}{(n+1)(2n+1)} - \cdots.$$

By the Cauchy ratio test, S_1 is absolutely convergent for $n \neq 0, -1, -1/2, -1/3 \cdots$. Hence S_1 and

$$1 - S_1 = \frac{1}{n+1} - \frac{1}{(n+1)(2n+1)} + \cdots$$

may be added term by term. This gives the series (1) on one side and $S_1+(1-S_1)=1$ on the other.

For (2) we take the series

$$S_2 = 3 - \frac{4}{n+1} + \frac{5}{(n+1)(2n+1)} - \cdots$$

and form (2) by adding S_2 to $(3-S_2)$ term by term. Since both S_2 and $(3-S_2)$ are seen to be absolutely convergent, the sum of (2) is $S_2+(3-S_2)=3$.

Also solved by C. B. Barker, Joshua Barlaz, Daniel Block, R. A. Bradley, Paul Brock, D. H. Browne, G. Y. Cherlin, A. B. Farnell, H. E. Fettis, H. Kaufman, Karl Itkin, J. F. Locke, Lowell Schoenfeld, F. C. Smith, F. Underwood, and the Proposer.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Probit Analysis: A Statistical Treatment of the Sigmoid Response Curve. By D. J. Finney. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 13+256 pages. $3.75.

Biological assay in general is the measurement of the potency of some stimulus by the reaction it produces in living material. This kind of measurement has been used for many years in assessing the potency of vitamins, hormones, toxicants and drugs of all kinds. One type of assay which has been widely used, especially in toxicological studies, is that depending on quantal, or all-or-none response. The most obvious example of this kind of response is death. Mr. Finney has written a book about the statistical treatment of quantal assay data by means of probit analysis.

The book is arranged in ten chapters and two appendices. In the first few chapters the technique of probit analysis is introduced in its simplest form. The following chapters discuss more complicated analyses and introduce the recent developments of the technique, many of which have been due to Mr. Finney. The first appendix gives a detailed computational scheme for probit analysis and the second a brief outline of the mathematical theory of the probit method. Seven tables which will lessen the computing time for a probit analysis complete the book.

The viewpoint of the individual, untrained in statistics, who has to analyze quantal assay data is maintained throughout. Although this is the viewpoint that is emphasized, enough of the theory is given at regular intervals to enable the trained statistician to follow the theoretical basis for the methods. Examples with detailed explanations and discussion are given to illustrate each new method or variation of a method. The examples are well chosen, largely from insecticide test data.

In places the explanations of the computations seem unnecessarily detailed. This, however, is probably not a serious objection, at least from the viewpoint of workers in the applied field. One finishes reading the book with the feeling that it is a worthy companion to books by other members of the Rothamsted group of statistical workers.

S. LEE CRUMP

College Algebra. By F. S. Nowlan. New York, McGraw-Hill Book Co., Inc., 1947. 14+371 pages. $3.00.

The preface of this book states, "The main purpose in preparing the present text has been the production of an algebra text mathematically sound and at the

same time easily understood. These objectives are not incompatible and the author believes that for an average student the most difficult treatment is one that is deficient in logic and incomplete or inexact in its statement of principles." The author has been quite successful in producing a text which is mathematically sound. However, he is subject to such lapses as speaking of the addition and multiplication of equations, and a statement on page 41 implies that subtraction is an associative operation. The reviewer has some doubts as to the ease with which an average student will understand the text. These doubts arise from the large number of precisely stated definitions, the multitude of theorems and corollaries with detailed proofs, and the wide variety of methods for solving some types of problems. This feature of the book, which makes it dull reading from a student's point of view, will make it extremely valuable as a reference book for a person who is new at teaching college algebra.

The subject matter is developed on the assumption that the student already understands the elementary properties of (real) positive numbers and zero and their laws of combination. This is presumably the justification for the omission of a number of topics, such as the decimal and other scales of notation, the irrationality of such numbers as $\sqrt{2}$, approximate numbers and their use in computation. The reviewer's experience leads him to believe that most students of college algebra do not have this postulated understanding.

The multitude of topics usually included in a college algebra text to provide a wide choice of subject matter appears in this book. The one notable exception is the solution of inequalities.

There are some minor points which deserve mention. Equations and their solutions are presented in a praiseworthy manner. The comparison method for solving two simultaneous linear equations seems superfluous. Graphical methods and much of the terminology of analytic geometry are used to solve equations and to clarify the explanation of the process. The examples which illustrate the various methods for solving equations regularly include a check by substitution and set a good example for the student. Although the function concept appears early, functional notation is not used until the chapter on theory of equations. A proportion is defined in terms of a proportionality constant and not as an equality of ratios. This makes possible the inclusion of zero among the terms of a proportion. The discussion of logarithms is marred by the use of a bar to indicate a negative characteristic. (A misprint on page 165 results in the equation $\log 0.256 = 1.4082$). An earlier appearance of a principle for addition would improve the discussion of permutations, combinations, and probability.

The most novel and outstanding feature of this book is the introduction of complex numbers. The symbol $[a, b]$, with a and b real numbers, is called a complex. The complexes $[a, b]$ and $[c, d]$ are equal if, and only if, $a=c$ and $b=d$. The rational operations on complexes are defined as

$$[a, b] + [c, d] = [a + c, b + d],$$
$$[a, b] - [c, d] = [a - c, b - d],$$

$$[a, b] \cdot [c, d] = [ac - bd, ad + bc],$$

$$[a, b] \div [c, d] = \left[\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right], \qquad [c, d] \neq [0, 0].$$

It is then shown that the special complexes $[x, 0]$ behave just like real numbers and these entities are identified. It is then shown that the complex $[0, 1]$ is a square root of $[-1, 0]$, so that the system of complexes provides square roots for negative numbers. Complexes are renamed complex numbers and are endowed with a rectangular form $x+iy$ and a polar form $\rho(\cos\theta+i\sin\theta)$. Although it is surprising that this procedure has not already appeared in a book of this type, it is gratifying to find it so well presented here. A more conventional approach to complex numbers is also provided.

The book contains a four place table of logarithms, interest and annuity tables, a mortality table, and a too brief table of powers and roots of integers. The problems are carefully graduated and are such as to provide a challenge for most students. There is a lack of simple exercises for rapid drill which are useful in many connections to give a student confidence.

All in all, it is a book with a broad field of usefulness and with enough novel and unusual features to be rewarding to anyone who looks into it carefully.

R. W. WAGNER

Intermediate Algebra. By J. R. Britton and L. C. Snively. New York, Rinehart and Company, 1947. 9+337 pages. $2.00.

Entitled "Intermediate," this text leads the student from the fundamental ideas of arithmetic (in a fifteen-page chapter), through fractions, exponents, linear and quadratic equations, common logarithms, as far as progressions and the binomial theorem. Tables of powers (exponents -1, 2, 1/2, 3, 1/3) and of five-place common logarithms with proportional parts constitute the appendix, which is followed by the answers to the odd-numbered problems.

The engineering viewpoint predominates. Exercises are numerous, including occasional review lists. The word problems expose an unusually wide variety of applications and constitute one of the outstanding features of the text. Graphing and the function concept receive considerable attention, more than might reasonably be expected in an intermediate text. Another pleasing characteristic is the insistence on an adequate check of the solution of a problem.

In presenting explanations and in solving illustrative examples, the authors have earnestly striven to write clearly and amply, avoiding the pitfall of excessive brevity so often a complaint of students. Only seldom do the authors relax the usual carefulness and allow a looseness of phraseology or an inconsistency of statements. The publishers have coöperated in offering a legible, attractive format and in nearly freeing the text from typographical error.

R. A. GOOD

NEW BOOKS RECEIVED

Table des Solutions de la Congruence $x^4+1\equiv 0 \pmod{p}$ *pour* $500\,000<p<6\,000\,000$. By A. Gloden. Luxembourg, 1947. 12 pages.

Liste des Formes Linéaires des Nombres dont le Carré se Termine dans le Système Décimal par une Tranche donné de 4 Chiffres. By A. Gloden. Luxembourg, 1947. 15 pages.

Table des Bicarrés N^4 *pour* $3001\leqq N\leqq 5000$. By A. Gloden. Luxembourg, 1947. 17 pages.

Tables of the Bessel Functions of the First Kind of Orders Thirteen, Fourteen, and Fifteen. Prepared by the Staff of the Computation Laboratory of Harvard University. Cambridge, Harvard University Press, 1947. $10.00.

Tables of the Bessel Functions of the First Kind of Orders Ten, Eleven, and Twelve. Prepared by the Staff of the Computation Laboratory of Harvard University. Cambridge, Harvard University Press, 1947. $10.00.

Nomography. By A. S. Levens. New York, John Wiley and Sons, Inc., 1948. 8+176 pages. $3.00.

Problème Général de la Stabilité du Mouvement. By A. Liapounoff. Translated from the Russian by E. Davaux. (Annals of Mathematics Studies, No. 17). Princeton University Press, 1947. 447 pages. $3.50.

Toward General Education. By E. J. McGrath and Others. New York, The Macmillan Company, 1948. 7+224 pages. $3.00.

Analytic Geometry. by D. S. Nathan and Olaf Helmer. New York, Prentice-Hall, Inc., 1947. 10+402 pages. $3.50.

College Algebra. By H. A. Simmons. New York, The Macmillan Company, 1948. 8+619 pages. $4.00.

Modern Mathematics for Everybody. By Duane Studley. Published by the Author, 1947. 84 pages.

The Atom. Third Edition. By G. Thomson. Oxford University Press, 1948. 196 pages. $2.00.

Fundamentals of Business Mathematics. By W. R. Van Voorhis and C. W. Topp. New York, Prentice-Hall, Inc., 1948. 8+454 pages $3.75.

A Handbook on Curves and their Properties. By R. C. Yates. Ann Arbor, Edwards, 1947. 10+245 pages.

ANSWER TO THE SECOND TEASER, PAGE 342

My answer is that from the data no conclusion can be reached, as to when the knife reaches the far edge of the table. Let s ft. be the distance of the knife from the edge where it started after t seconds. The value of the dependent variable t is given for a discrete set of values of the variable s. The value $s=4$ is not included in this discrete set, and hence we do not know what t is when $s=4$. If we add further hypotheses, to the effect that t is a monotonic increasing function of s on the range $0\leqq s\leqq 4$, and is continuous at $s=4$, we might conclude that $t=2$ when $s=4$. Otherwise who knows whether t makes finite jumps!

E. J. MOULTON

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Kappa Mu Epsilon, Albion College

The members of the Michigan Alpha Chapter of *Kappa Mu Epsilon* presented the following papers during the year:

Bridging the gap between high school mathematics and college mathematics, by Audrey Shuett

Some mathematical riddles and puzzles, by Faye Engstrom

The field of mathematics, by Russell Meier

Mathematics as set up at universities in Europe during the war, by Olive Conway

What the teacher of mathematics should be, by Erwin Weber

The original calculating machine, by William Hopkins

The origin of the dollar sign, by George Kawano

Parallel lines by Amy Thomas

Accuracy of linear interpolation, by John Nixon.

In addition to the regular club reports an outside speaker, Mr. Gable, gave a talk on *The practical side of the field of engineering*. At each meeting, response to roll call was made by giving formulas from certain fields, such as: Analytic Geometry, Plane Geometry, and so on.

At the fall meeting of the Michigan Section of the Mathematical Association of America a paper, *Development of Mathematics in Scotland* 1717–1838, was read by Shirley Searles.

According to tradition, there also was a joint-picnic in May with the Chemistry Club and the Physics Club.

The officers who served during this year were: President, Audrey Shuett; Vice-president, George Kawano; Secretary-treasurer, Shirley Searles. The officers elected for 1947–48 are: President, Robert Olsen; Vice-president, Faye Engstrom; Secretary-treasurer, George Karres.

Pi Mu Epsilon, Pennsylvania State College

A mathematics club sponsored by the Pennsylvania Delta Chapter of *Pi Mu Epsilon* met regularly during the latter part of the academic year. At the first meeting of the year, some moving pictures on *Vector analysis* and on *Aerodynamics* were shown, and at subsequent meetings the following student talks were given:

The concept of infinity, by Michael Fatula

The four-color problem, by Jerome Berliner

The planimeter, by C. H. Seaton

Cones and conics, by Joseph Konhauser

Congruences, by Kenneth Harshbarger.

The annual mathematics contest was won by Joseph Konhauser.

The officers for 1946–47 were: President, J. C. Dillard; Vice-president, Martin Lennig; Secretary, Vivian Klein. The officers elected for 1947–48 are: President, Joseph Konhauser; Vice-president, Robert Foote; Secretary, Alice Shade Scollon.

Delta-Y Club, D'Youville College

The *Delta-Y Club* activities for the year 1946–47 had for its theme the discussion of unusual problems. Each of the six meetings was devoted to the presentation of a problem as suggested by two of the members.

The program consisted of:

An analytical proof for the circle of Apollonius, by Mary Ellen Keiran

Development of a formula for the relationship of two people, by Agnes Allard and Jane Deckop

The theorem of Pythagoras and its relation to an analogous cubic equation, by Helen Hand and Joan Geitner

A problem of navigation, by Shirley Heary and Ann Healy

Ptolemy's theorem, by Mary Ellen Keiran and Alice Staebell.

As a final meeting, a dinner followed by a mathematical quiz was held on May 14. Prospective members were invited.

Officers for the year 1946–47 are: President, Helen Hand; Secretary, Joan Geitner.

Kappa Mu Epsilon, Mount St. Scholastica College

The *Kansas Gamma* Chapter had for the theme of its year's activities *An integrated approach to science*. Special stress was laid on the need for scholarship among scientists. A panel discussion was held during the year, the subject being *The compatability of religion and science*. Ronald Knox's book, *God and the atom*, was reviewed at one of the meetings.

The following papers written by members of the chapter were presented at the meetings:

A plea for non-isolationism in Mathematics, by Victoria Fritton

An integral approach to Mathematics, by Frances Walsh

The Hellenic schools and Mathematics, by Mary Jo Schieberl

The nature of mathematical reasoning, by Mary Jane Fox.

The chapter has continued the quarterly publication of the *Exponent*, the chapter paper. A pledge manual containing the history and other helpful information about the fraternity has been prepared by several chapter members.

Following the initiation of twelve new members in May, a formal dinner was served. The same evening the Hypatian award was formally presented to members who demonstrated outstanding scholarship and leadership in the form of contribution to the field of mathematics.

Officers for 1946–47 were: President, May Jane Fox; Vice-president, Joanne

Shuey; Secretary, Victoria Fritton; Treasurer, Jean Moran; Publicity Chairman, Frances Knightley; Chapter Musician, Elizabeth Gulde; Corresponding Secretary and Faculty Sponsor, Sister Helen Sullivan, O.S.B.

Officers for 1947–48: President, Victoria Fritton; Vice-president, Joanne Shuey; Secretary, Jean Moran; Treasurer, Frances Knightley; Publicity Chairman, Gloria Jaskoviak; Chapter Musician, Mary Alice Weir; Corresponding Secretary and Faculty Sponsor, Sister Helen Sullivan, O.S.B.

Mathematics Club, Brown University

The thirty-third annual program of the *Mathematics Club* of Brown University consisted of six regular meetings with talks by faculty and students. The following papers were presented:

Predicting eclipses, by Professor C. H. Smiley of the Department of Astronomy

Lewis Carroll, author and mathematician, by Ann Henry

Newton and the Calculus, by Michel Antone

Continued fractions, by Sybil Tanenbaum

Leibniz and the Calculus, by Kenneth Crowe

Whitney's ham sandwich problem, by Professor Maurice Heins of the Department of Mathematics

Sylvester and invariance, by Ann Kooker

Summation of finite series, by Kenneth Casey

Your chance to lose, by John Houston

Diophantine equations, by John Hynes.

Each of the regular meetings concluded with a social and discussion period at which light refreshments were served. This period provided an opportunity for students and faculty to become better acquainted. A club picnic was held on May 17 as the concluding group activity for the current year.

The total student membership for the year was sixty-five. Fifteen faculty members participated in the activities.

The committee on Program and Arrangements for the year 1946–47 was composed of: Stephen Prager, Chairman, Philip Bray, Doris Hackett, LeRoy Peckham, Jr., and Sybil Tanenbaum. The members of the committee for the year 1947–48 are: LeRoy Peckham, Jr., Chairman, Thelma Andrews, Philip Bray, Kenneth Crowe, Ann Henry, Robert Kotlen, Eleanor Mansfield, Donald Nolan, and Howard G. Smith, Dr. Western is the faculty representative.

Mathematics Club, Woodrow Wilson Branch of Chicago City Junior College

During the war, most of the activities of the *Mathematics Club* at the Woodrow Wilson Branch of the Chicago City Junior College were suspended and have not been resumed. However, thirteen neighboring high schools participated in the sixth annual *Mathematics Tournament* on April 21, 1947. The purpose of these tournaments is to stimulate interest in mathematics in those high schools which contribute students to the Wilson Branch.

Problems were run off in relays of three each for eight relays and prizes were awarded to the four highest schools and to the six highest individuals. The four contestants sent from each high school must have had at least one and one-half years of high school algebra and one year of plane geometry and were generally selected by preliminary contests. After the tournament, tea was served to the students and their teachers while the prizes were being awarded.

Mathematics Club, University of Kansas

The *Mathematics Club* of the University of Kansas presented the following programs:

Flatland, by Samuel Kneale
Mathematics of double stars, by Dr. N. W. Storer
Sketches of famous mathematicians, by Merilyn Whiting
Calculation of leap year, by William Stoner
Algebraic numbers, by Dr. Robert Schatten
Nine-point circle, by June Hull
Differential analyzers, by Dale Rummer
Pythagorean numbers, by Dr. G. W. Smith
Some geometric considerations of molecular structure, by James Mordy.

The annual picnic was held in May.

Officers for 1946–47 were: President, Louis Harkleroad; Vice-president, Dale Rummer; Secretary-treasurer, June Hull.

The officers elected for the year 1947–48 are: President, Marietta Witherspoon; Vice-president, Joseph Hull; Secretary-treasurer, John Michener.

Mathematics Club, Westminster College

Delta Nabla, the Westminster College *Mathematics Club* held five regular meetings in 1946–47. The aims, purposes, and history of *Delta Nabla* were discussed at the formal initiation dinner. Each new member then reviewed the mathematician's life whose name he chose to bear during membership in the club. After the pledge was taken the group went to a moving picture.

Robert Nesbitt received the annual Freshman Award presented for outstanding scholastic achievement.

Wilfred Gibson gave a report of the activities of the club at the Annual Spring Meeting of the American Mathematical Society held in Chicago, April 26, 1947.

The officers for the year 1946–47 were: President, Edna Friggle; Vice-president, Wilfred Gibson; Secretary-treasurer, Mary Elizabeth Clark.

Officers of 1947–48 are: President, Amelia Duff; Vice-president, Edna Friggle; Secretary-treasurer, Mary Elizabeth Clark.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

ANNOUNCEMENT OF NAVY DEPARTMENT JOINT BOARD

The Navy Department Joint Board of U. S. Civil Service Examiners has announced that it is expanding three comparatively new, permanent laboratories in California. The program of the Naval Ordnance Test Station, China Lake, California, involves research, development and test work with ordnance equipment and explosives. The Navy Electronics Laboratory, San Diego, California, is concerned with research, testing and development of electronic control devices, detection equipment, instrumentation equipment and training aids. The activities of the Naval Air Missile Test Center, Point Mugu, California pertain to flight and laboratory testing and evaluation of guided missiles and their components.

The Board wishes to recruit qualified personnel for these laboratories. Examinations are now open in the following fields: Chemist, Mathematician, Metallurgist, Meteorologist, Physicist, Statistician, Scientific Research Administrator and Scientific Staff Assistant. Salaries range from $3397 to $9975 per annum.

Further information may be obtained from the Navy Department Joint Board of U. S. Civil Service Examiners, 1030 East Green Street, Pasadena 1, California.

THE CANADIAN MATHEMATICAL CONGRESS

The Canadian Mathematical Congress, which was organized in 1945, will hold a second national congress and seminar in 1949 at the University of British Columbia.

Any full time member of the mathematical teaching staff of a Canadian university, or any person who was a member of the Congress of 1945, or of the seminar of 1947, and any other person sponsored by a member of the Congress may become a member of the Congress on application and payment of the membership fee and acceptance by the Executive Committee. The fee for membership is $2.00 per annum.

Application for membership should be sent to: Secretary, Canadian Mathematical Congress, Engineering Building, McGill University, Montreal, Canada.

CANADIAN JOURNAL OF MATHEMATICS

The Canadian Mathematical Congress has announced the publication of a new journal, *Canadian Journal of Mathematics*, beginning January, 1949.

Manuscripts for publication in the *Journal* should be sent to the Editor-in-Chief, H. S. M. Coxeter, University of Toronto. Every paper should contain an introduction summarizing the results as far as possible in such a way as to be

understood by the non-expert.

The *Journal* will be published quarterly. The price per volume of four numbers is $6.00. This is reduced to $3.00 for members of the Mathematical Association of America. Subscriptions should be sent to: Managing Editor, G. de B. Robinson, University of Toronto, Toronto, Canada.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1948:

Syracuse University. June 21 to July 31: vector analysis, differential equations, mathematics of statistics, probability, seminar, topics in partial differential equations, functions of a complex variable, mathematical methods of fluid dynamics. August 2 to September 4: introduction to modern algebra, advanced calculus, mathematics of statistics, advanced mathematics for engineers and physicists, probability, seminar, advanced functions of a complex variable, differential geometry.

University of California. Session II. August 2 to September 11: Professor Zariski, introduction to the arithmetic theory of algebraic varieties.

West Virginia University. Second term: Professor Stewart, modern synthetic geometry; Professor Vehse, theory of determinants and analytic geometry of space, calculus of variations; Professor Peters, theory of numbers; Professor Davis, Cremona transformations.

MATHEMATICS INSTITUTE FOR TEACHERS

A Mathematics Institute for Teachers will be held at Duke University on August 9–20, 1948. The general theme of the Institute is Mathematics at Work. The program includes lectures pertaining to the applications of mathematics given by representatives of various industries. Study groups will be organized to consider topics of interest to teachers of secondary-school and college mathematics. Programs may be obtained from Professor W. W. Rankin, Duke University, Durham, North Carolina.

PERSONAL ITEMS

The officers of Section V_1, Philosophy of Mathematics, of the Tenth International Congress of Philosophy are as follows: President, L. E. J. Brouwer; Vice-President, A. Heyting. Professor M. H. Stone and Professor H. B. Curry will be speakers. The Congress will meet in Amsterdam, Holland on August 11–18, 1948.

Assistant Professor Richard Bellman of Princeton University has been appointed to an associate professorship at Stanford University, effective September, 1948.

Associate Professor W. M. Borgman, Jr. of Wayne University has been appointed Assistant Dean of Administration.

Professor T. G. Cowling of the University College of North Wales has been appointed to a professorship at the University of Leeds.

Dr. H. D. Huskey, who was engaged in research at the British National Physical Laboratories during the year 1947, is now Chief of the Machine Development Laboratory of the National Bureau of Standards.

Dr. G. R. MacLane of Harvard University has been appointed to an assistant professorship at Rice Institute.

Professor G. M. Magee of the University of Western Ontario has accepted a position as visiting professor at Wayne University for the summer session.

Dr. Rufus Oldenburger, formerly professor of mathematics at Illinois Institute of Technology, has been appointed Chairman of the Mathematics Department of De Paul University.

Dr. C. Y. Panc of the University of Marseille has been appointed lecturer at the University of Cape Town.

Professor Joseph Pierce of Atlanta University has been appointed to an associate professorship at Wayne University.

Professor Joseph Polley of Wabash College will be a visiting professor at Wayne University for the summer session.

Dr. R. D. Schafer of the Institute for Advanced Study has been appointed to an assistant professorship at the University of Pennsylvania.

Dr. H. W. E. Schwerdtfeger of the University of Adelaide has been appointed to the position of senior lecturer at the University of Melbourne.

Professor Jabir Shibli, formerly professor of mathematics at Pennsylvania State College, has been appointed to a professorship at Presbyterian College, Clinton, South Carolina.

Associate Professor Max Shiffman of New York University has been appointed to a professorship at Stanford University, effective September, 1948.

Dr. C. V. L. Smith is now Head of the Computing Machine Section of the Mathematics Branch of the Office of Naval Research.

Miss Mary V. Sunseri of Stanford University has been promoted to an acting assistant professorship, effective September, 1948.

Assistant Professor P. M. Whitman of Tufts College has accepted a position at the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Professor G. F. Woodson, Jr. is now Head of the Department of Mathematics of the College of Education and Industrial Arts, Wilberforce, Ohio.

Wayne University announces the appointment of Mr. Jack Patterson as instructor.

Associate Professor Emeritus James K. Whittemore of Yale University died March 22, 1948 at the age of seventy-three years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following eighty-eight persons have been elected to membership by the Board of Governors on applications duly certified:

J. J. BARRON, Ph.D. (Wisconsin) Professor, Marshall College, Huntington, W. Va.

BARBARA B. BETTS, A.M. (Radcliffe) Editor, Mathematics Department, D. C. Heath and Company, Boston, Mass.

LOLLIE BELLE BIENVENU, M.S. (Louisiana State) Assistant, Louisiana State University, Baton Rouge, La.

L. B. BOND, B.S. (University of Washington) Instructor, Seattle College, Seattle, Wash.

R. L. BROOKS, B.S. (West Virginia Wesleyan) Graduate Student, University of North Carolina, Durham, N. C.

K. E. BROWN, Ph.D. (Columbia) Head of Department, Wagner College, Staten Island, N. Y.

J. W. BUTLER, Associate Engineer, Clinton National Laboratories, Oak Ridge, Tenn.

THOMAS CHERBAS, Student, Drexel Institute Evening School, Philadelphia, Pa.

G. Y. CHERLIN, B.S. (Rutgers) Asst. Instructor, Rutgers University, New Brunswick, N. J.

B. A. CHIAPPINELLI, Student, University of California at Los Angeles, Calif.

THERESA M. CHIAVERINI, B.A. (Michigan State) Instructor, University of Detroit, Mich.

H. B. COLEMAN, M.S. (Michigan) Assistant, University of Michigan, Ann Arbor, Mich.

A. E. COOK, A.B. (Georgetown College) Asst. Professor, Georgetown College, Georgetown, Ky.

P. R. CULWELL, M.A. (Texas) Asst. Professor, Trinity University, San Antonio, Texas

W. W. DOLAN, Ph.D. (Oklahoma) Asst. Professor, University of Oklahoma, Norman, Okla.

E. B. EILERTSEN, M.A. (California) Instructor, City College of San Francisco, Calif.

F. N. FISCH, M.A. (Colorado State) Asst. Professor, Colorado State College, Greeley, Colo.

W. T. FISHBACK, A.M. (Harvard) Teaching Fellow, Harvard University, Cambridge, Mass.

T. A. GAFFNEY, M.S. (California Institute of Technology) City College of San Francisco, Calif.

E. I. Gale, A.M. (Columbia) Asst. Professor, University of New Brunswick, Fredericton, N. B.

M. J. GERDES, President, Wayne Metal Finishing Company, Brooklyn, N. Y.

B. K. GOLD, JR., M.A. (U.C.L.A.) Instructor, Los Angeles City College, Calif.

DAVID GORDON, M.A. (Columbia) Chairman of Department, Walton High School, Bronx, N. Y.

R. P. GRAHAM, Electrical Engineer, Evanston, Ill.

J. S. GRIFFIN, JR., Student, Alabama Polytechnic Institute, Auburn, Ala.

A. G. HANSEN, M.S. (Purdue) Instructor, Purdue University, West Lafayette, Ind.

C. C. C. HARDING, B.A. (Columbia) Asst. Manager, General Section, Treasurer Division, E. I. DuPont DeNemours and Co., Wilmington, Del.

M. B. HASLAM, Student, University of Buffalo, Buffalo, N. Y.

A. H. HILL, M.S. (Wisconsin) Professor, North Dakota Agricultural College, Fargo, N. D.

A. T. HIND, JR., M.A. (Emory) Instructor, Clemson College, Clemson, S. C.

J. M. HOWELL, M.A. (U.C.L.A.) Instructor, Los Angeles City College, Calif.

KARL ITKIN, B.S. (Cooper Union) Electrical Engineer, Federal Power Commission, Washington, D. C.

S. B. JACKSON, Ph.D. (Harvard) Asso. Professor, University of Maryland, College Park, Md.

VIVIAN V. JOHNSTON, B.S. (Geneva) Instructor, Tusculum College, Greeneville, Tenn.

M. S. KLAMKIN, M.S. (Brooklyn Polytechnic Institute) Teaching Fellow, Carnegie Institute of Technology, Pittsburgh, Pa.

REV. BONAVENTURE KNAEBEL, M.S. (Catholic University) Instructor, St. Meinrad's College, St. Meinrad, Ind.

J. C. KNIPP, Ph.D. (Pittsburgh) Asso. Professor, University of Pittsburgh, Pittsburgh, Pa.

F. T. KOCHER, JR., B.S. (Bloomsburg State) Instructor, Pennsylvania State College Undergraduate Center, DuBois, Pa.

C. E. LANGENHOP, M.S. (Iowa State) Instructor, Iowa State College, Ames, Iowa

J. C. LANZ, Sc.M. (Brown) Teacher, Norristown School Board, Norristown, Pa.

C. H. LINDAHL, M.S. (Colorado) Asst. Professor, Iowa State College, Ames, Iowa

MARTIN MALTENFORT, A.M. (Montclair State) Instructor, Manhattan College, New York, N. Y.

R. W. MARSH, A.B. (American University) Graduate Student, George Washington University, Washington, D. C.

C. F. MARTIN, B.S. (U. S. Naval Academy) Instructor, University of South Carolina, Columbia, S. C.

W. J. MCCALLION, M.A. (McMaster) Lecturer, McMaster University, Hamilton, Ont.

L. H. MILLER, Ph.D. (Ohio State) Asst. Professor, Ohio State University, Columbus, Ohio

W. G. MILLER, M.A. (Florida) Asst. Professor, Clemson College, Clemson, S. C.

ABIGAIL M. MOSEY, M.A. (Syracuse) Instructor, Hobart College, Geneva, N. Y.

B. H. MOUNT, JR., M.S. (Princeton) Asst. Professor, University of Pittsburgh, Pittsburgh, Pa.

REV. J. P. MURRAY, S.J., M.A. (Boston College) Instructor, Fairfield University, Fairfield, Conn.

EUGENE ODIN, M.E. (Cornell) Senior Project Engineer, Arma Corporation, Brooklyn, N. Y.

E. F. ORMSBY, M.S. (Syracuse) Instructor, Union College, Schenectady, N. Y.

REV. E. F. O'SHEA, S.J., A.B. (Woodstock College) Instructor, University of Scranton, Scranton, Pa.

T. G. OSTROM, Ph.D. (Minnesota) Asst. Professor, Montana State University, Missoula, Mont.

W. D. PEEPLES, JR., B.S. (Howard) Graduate Asst., University of Wisconsin, Madison, Wis.

LILLIAN G. PERKINS, B.S. (South Carolina) Instructor, University of South Carolina, Columbia, S. C.

O. L. PHILLIPS, M.A. (North Texas State) Instructor, Louisiana State University, Baton Rouge, La.

C. F. PINZKA, Student, Rutgers University, New Brunswick, N. J.

J. T. PITTS, Student, Furman University, Greenville, S. C.

GERTRUDE V. PRATT, A.M. (Michigan) Central Michigan College of Education, Mt. Pleasant, Mich.

J. J. QUINN, M.S. (New York) Instructor, Bayonne Junior College, Bayonne, N. J.

ALICE B. RABON, M.Ed. (South Carolina) Instructor, University of South Carolina, Columbia, S. C.

G. P. RIGSBY, Student, California Institute of Technology, Pasadena, Calif.

R. M. ROBINSON, M.S. (Drake) Instructor, Iowa State College, Ames, Iowa

LOUIS SACKS, M.S. (Carnegie) Instructor, Carnegie Institute of Technology, Pittsburgh, Pa.

F. L. SANDER, Student, Michigan State College, East Lansing, Mich.

L. E. SCHAEFER, M.A. (Michigan State) Instructor, General Motors Institute, Flint, Mich.

N. C. SCHOLOMITI, B.S. (New York) Instructor, De Paul University, Chicago, Ill.

R. C. SEBER, B.A. (Coe) Graduate Asst., State University of Iowa., Iowa City, Iowa

E. B. SHANKS, Ph.D. (Illinois) Asst. Professor, Vanderbilt University, Nashville, Tenn.

B. I. SHOESMITH, B.S. (Chicago) Instructor, Illinois Institute of Technology, Chicago, Ill.

SISTER MARIE LORETTA, O.P., M.A. (Catholic University) Instructor, Siena Heights College, Adrian, Mich.

A. H. SMITH, Ph.D. (Brown) Asso. Professor, Purdue University, West Lafayette, Ind.

R. E. SMITH, M.A. (Pittsburgh) Asst. Professor, Duquesne University, Pittsburgh, Pa.

L. A. SNIDER, Student, George Washington University, Washington, D. C.

R. L. SNIDER, B.S. (Missouri) Instructor, Kemper Military School, Boonville, Mo.

E. V. SOMERS, M.S. (Pittsburgh) Research Engineer, Westinghouse Electric Corporation, East Pittsburgh, Pa.

ISAAY STEMPNITZKY, S.M. (Massachusetts) Research Assistant, Massachusetts Institute of Technology, Cambridge, Mass.

RUTH K. SWANSON, M.S. (Oklahoma) Instructor, Friends University, Wichita, Kan.

EDWIN TABOR, A.B. (California) 2319½ Haste St., Berkeley, Calif.

KAIDY TAN, Instructor, Chip-Bee Institute, Amoy, China

FLORENCE G. TETREAULT, B.S. (Detroit) Instructor, University of Detroit, Detroit Mich.

S. L. THOMPSON, M.A. (Michigan) Asst. Professor, Alabama Polytechnic Institute, Auburn, Ala.

LONA L. TURNER, M.A. (Michigan) Student, University of Chicago, Chicago, Ill.

BERNICE L. WARR, Student, Skidmore College, Saratoga Springs, N. Y.

L. B. WILLIAMS, S.M. (Chicago) Asst. Professor, Reed College, Portland, Ore.

R. W. YOUNG (Indiana), Instructor, Lehigh University, Bethlehem, Pa.

G. C. ZADER, B.S. (Davis-Elkins) Instructor, The Citadel, Charleston, S. C.

NOVEMBER MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The twentieth meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Carnegie Institute of Technology, Pittsburgh, Pennsylvania, on Saturday, November 22, 1947. Professor J. B. Rosenbach presided at the morning and afternoon sessions.

The attendance was one hundred sixty-two, including the following thirty-nine members of the Association: O. F. H. Bert, J. O. Blumberg, R. C. Briant, A. M. Bryson, Helen Calkins, W. E. Cleland, H. L. Dorwart, Benjamin Epstein, F. A. Foraker, Tomlinson Fort, E. T. Frankel, Beatrice L. Hagen, Helen Harmon, B. P. Hoover, H. L. Krall, Elizabeth L. Lahti, H. R. Leifer, C. G. Maple, A. W. McGaughey, E. W. Montroll, David Moskovitz, L. T. Moston, C. E. Mullan, J. H. Neelley, E. G. Olds, Morris Ostrofsky, F. W. Owens, Helen B. Owens, J. B. Rosenbach, Edward Saibel, H. C. Shaub, R. E. Smith, F. H. Steen, Helen F. Story, J. L. Synge, J. S. Taylor, Margaret O. Taylor, C. H. Vehse, E. A. Whitman.

The officers elected at the business meeting were as follows: Chairman, J. B. Rosenbach, Carnegie Institute of Technology; Secretary-Treasurer, E. W. Montroll, University of Pittsubrgh; additional members of the Executive Committee, F. H. Steen, Allegheny College, H. L. Krall, Pennsylvania State College. The next meeting will be held on May 8, 1948, at Pennsylvania State College, State College, Pennsylvania.

The program consisted of the following papers:

1. *The United States Army experiment in higher education*, by Professor J. H. Neelley, Carnegie Institute of Technology.

Professor Neelley described the operation and achievements of Biarritz American University on whose mathematics staff he served.

2. *Averages over functions of characterisitic values of linear operators*, by Pro-

fessor E. W. Montroll, University of Pittsburgh.

Crystalline solids and polyatomic molecules in the gas phase can be represented by a set of coupled harmonic oscillators. The thermodynamic properties of such substances are determined theoretically by averaging functions of the frequencies of the normal modes of vibration of the equivalent oscillators over all the normal modes. Because of the great number of degrees of freedom of a solid, the exact calculation of frequencies of normal modes is impossible. It was shown that without too much difficulty one can obtain the traces of powers of the matrices of the secular determinants. These traces in turn give the moments of the distribution function of the normal modes. From these moments one can estimate the thermodynamic averages by mechanical quadratures. The upper bound to the error resulting from the use of a specified number of moments is given by an expression due to Markoff. Analogous procedures can be used to determine averages over characteristic values of linear operators other than matrices.

3. *Boundary problems for difference equations*, by Professor Tomlinson Fort, University of Georgia.

Professor Fort spoke on boundary problems for the linear difference equation with integral independent variable. The topic was introduced and motivated by a discussion of a weighted string vibrating in a plane with small vibrations. Known results for the equation of the second order were summarized, and brief reference to the equation of the nth order was made.

4. *Open convex surfaces which are rigid*, by Professor J. J. Stoker, New York University, introduced by the Secretary.

A two-dimensional surface in three-dimensional space is said to be rigid if it possesses as infinitesimal length-preserving deformations (with continuous third derivates) only rigid body motions. A surface is not rigid in the small in general, but it has been shown by Liebmann, Blaschke, and Weyl that a closed convex surface is, in its whole extent, rigid. It was shown that certain open convex surfaces are rigid, provided only that the infinitesimal deformations considered are bounded —a condition which is automatically fulfilled in the case of closed surfaces. The open surfaces considered belong to the class of complete surfaces, that is, surfaces on which every Cauchy sequence converges (in the sense of the Riemannian metric on the surface) to a point on the surface. Complete open convex surfaces in three-dimensional space can always be represented in the form $z=z(x, y)$ in suitably chosen Cartesian coördinates. It was shown that the following two classes of surfaces of this kind are rigid: (1) those in which $z(x, y)$ is defined over the entire xy-plane; and (2) all surfaces of revolution.

5. *Some recent trends in differential geometry*, by Professor Enrico Bompiani, University of Rome, introduced by the Secretary.

The theory of point transformations between two projective planes (or spaces) has recently attracted the attention of E. Kasner and J. DeCicco. A survey was given of some results obtained in Italy during the war by Dr. Villa and the speaker concerning the approximation of those transformations by birational or Cremona transformations, or, when that is not possible, by unirational transformations. Interest in the topic was accentuated by the development of some unexpected facts.

6. *Newtonian polynomials*, by Professor H. L. Krall, Pennsylvania State College.

Starting from Vandermonde's equation $(a+b)^{(n)}=\sum_{i=0}^{n}\binom{n}{i}a^{(i)}b^{(n-i)}$ various other formulas for Newtonian or factorial polynomials were developed. Some of these were used to obtain results concerning factorial moments.

7. *A mathematical approach to certain aspects of the breakage of solids*, by Dr. Benjamin Epstein, Carnegie Institute of Technology.

A mathematical model of certain breakage processes was constructed. The following assumptions were made: (a) any breakage process is composed of discrete steps; (b) there is a function $P_n(y)$, the probability of breakage of particles of size y in the nth step of the breakage process; and (c) there is a function $F(x, y)$, the distribution function of pieces of size $x \leqq y$, arising from the breakage of a unit mass of size y.

If, in particular, $P_n(y)$ is a constant p_n, independent of y (but possibly dependent on n), and $F(x, y)$ is such that the percentage ($<Ky$, $0 \leqq K \leqq 1$) of material arising from the breakage of a unit mass of size y is independent of y, then the distribution function $F_n(x)$ after n steps in the process is the distribution function of a product of $(n+1)$ independent random variables. Under minor restrictions this means that the size distributions are asymptotically logarithmico normal. This result may explain partially why fine grinding and crushing operations are often observed to yield logarithmico-normal distribution. Generalizations of the problem under other assumptions concerning $P_n(y)$ and $F(x, y)$ were briefly considered. Most of the results are contained in a paper in the December 1947 issue of the *Journal of the Franklin Institute*.

8. *An undergraduate program in mathematics*, by Professor Morris Ostrofsky, Duquesne University.

The undergraduate program in mathematics at Duquesne University serves the dual purpose of preparing some students for graduate work in mathematics, and training others who are in the fields of natural sciences. The first two years are identical for both groups. In the first semester of the freshman year a five hour course is offered. This course is intended to prepare students for elementary calculus and is designed to teach methods rather than mere operational techniques. It begins by developing the number system from positive integers, and includes the discussion of complex numbers, logarithms, algebraic equations, plane trigonometry, and some analytical geometry. Calculus begins the second semester of the freshman year. Integral and differential calculus are studied simultaneously. More analytical geometry is studied along with the calculus. During the first semester of the second year, calculus is continued. The second semester of the sophomore year is devoted to a course in elementary differential equations. Students majoring in mathematics, in their third year, are required to take a course in advanced calculus designed to serve as a good introduction to function theory. To round out their program they can take courses in analytical mechanics, algebra, integral equations and, possibly, topology. Science majors are offered additional courses in ordinary and partial differential equations, or a course at the level of advanced calculus but with emphasis on applications rather than rigor.

H. L. DORWART, *Secretary*

FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Maryland, College Park, Maryland, on December 6, 1947, with Dean E. J. Finan, Chairman of the Section, presiding at the morning and afternoon sessions.

Seventy persons attended the meeting, including the following thirty-eight members of the Association: N. H. Ball, W. Bleick, H. Campaigne, Randolph Church, C. R. Clark, A. Cohen, J. A. Duerksen, P. J. Federico, Anselm Fisher, E. E. Floyd, M. K. Fort, Jr., B. C. Getshell, Michael Goldberg, R. A. Good, Thomas Greville, G. A. Hedlund, M. A. Hyman, J. E. Ikenberry, Walter Jennings, Sidney Kaplan, A. E. Landry, Carol V. McCamman, E. J. McShane,

M. E. Meade, A. K. Mitchell, W. K. Morrill, T. W. Moore, A. B. Mewborn, J. M. Popow, O. J. Ramler, R. W. Rector, J. N. Rice, A. Tierney, C. C. Torrance, Eunice D. Schell, W. R. Utz, A. L. Whiteman, G. T. Whyburn.

The following papers were presented at the morning session:

1. *Homotheticity—an effective tool in proving certain types of theorems in geometry*, by Professor O. J. Ramler, Catholic University.

After defining homothetic rectilinear figures, the speaker proved the fundamental theorem on homothetic figures. This theorem states that lines joining corresponding points of homothetic figures are concurrent. It was used to prove that the centroid, orthocenter, circumcenter, and the center of the nine-point circle of a triangle are collinear points.

2. *Rotors in polygons*, by Dr. M. Goldberg, Bureau of Ordnance, Navy Department, Washington, D. C.

A rotor of a polygon was defined to be a closed convex curve which remains tangent to all sides of the fixed polygon during a complete rotation of the curve. Constructions for rotors were given, and several working models of them were exhibited.

3. *A trigonometric identity*, by Dr. A. L. Whiteman, Navy Department, Washington, D. C.

If $p=4n+3$ is prime, it is known that the set of integers $1, 2, 3, \cdots, (p-1)/2$ contains more quadratic residues of p than non-residues. The only known derivations of this result are transcendental. The author establishes a trigonometric identity which leads to the following two equivalent theorems. If $p=4n+3$ is prime, then $\sum_{m=1}^{p-1} \cot(\pi m^3/p)>0$ and $\sum_{m=1}^{(p-1)/2} [m^3/p] > (p-1)(p-5)/24$.

4. *The sound of a projectile moving at supersonic speed*, by Professors S. B. Jackson (introduced by the Secretary), and M. H. Martin, University of Maryland.

A projectile moving at supersonic velocity v in a plane sets up a sound wave whose wave front is the envelope of a family of circles with centers at the points of the trajectory. In the case of a projectile acted on by gravity and by a retardation $f(v)$, it was shown that the wave front turns its concave side toward the trajectory so that an observer in the plane of the motion hears the sound from a uniquely determined apparent source, provided that $f(v)$ is continuous, that $f(v)/v$ increases with v, and that the limiting velocity is less than the speed of sound.

5. *The first year course in college mathematics*, by Professor E. J. McShane, of the University of Virginia.

This address was given by invitation. The speaker gave an interesting account of the course in Freshman Mathematics required of Arts and Sciences students at the University of Virginia. The classes meet five times a week for two semesters. The first semester is devoted to algebra exclusively, beginning with elementary logic, and the derivation of the ordinary laws of algebra from a set of postulates. Emphasis is placed on the understanding and appreciation of a mathematical argument rather than the cultivation of mathematical technique and familiarity with mathematical formulae. The course becomes more conventional as the semester progresses; and by the end of the semester, the usual topics in algebra have been covered. The second semester is devoted to trigonometry and analytic geometry with a minimum of time spent on solution of triangles. The reduction of the conic sections to standard forms is omitted.

M. H. MARTIN, *Secretary*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Bryn Mawr College, Bryn Mawr, Pennsylvania, on Saturday, November 29, 1947. Professor C. J. Rees, Chairman of the Section, presided at the morning and afternoon sessions.

There were sixty present, including the following forty-one members of the Association: P. T. Bateman, T. A. Botts, H. W. Brinkmann, W. B. Campbell, P. A. Caris, V. F. Cowling, J. E. Davis, F. L. Dennis, Arnold Dresden, N. J. Fine, C. D. Firestone, J. R. Kline, V. V. Latshaw, Marguerite Lehr, M. Le Leiko, F. L. Manning, Clifford Marburger, E. A. McDougle, S. S. McNeary, A. E. Meder, Jr., Martin Moliver, Lillian Moore, F. D. Murnaghan, W. R. Murray, C. A. Nelson, J. C. Oxtoby, A. E. Pitcher, A. O. Qualley, Edward Rayher, G. E. Raynor, C. J. Rees, I. J. Schoenberg, L. L. Smail, A. W. Tucker, R. M. Walter, J. B. Walton, G. C. Webber, Anna Pell Wheeler, S. S. Wilks, H. M. Zerbe.

At the business meeting the following officers were elected for the coming year: Chairman, E. P. Starke, Rutgers University; Secretary, T. A. Botts, University of Delaware. The Program Committee for the next meeting will be: W. H. Gottschalk (Chairman), University of Pennsylvania, J. C. Oxtoby, Bryn Mawr College, and L. L. Smail, Lehigh University.

The program consisted of the following papers:

1. *On Walsh functions*, by Professor N. J. Fine, University of Pennsylvania.

The Walsh functions $\{\psi_n(x)\}$ are introduced as the completion of the Rademacher system. It is shown that the $\{\psi_n(x)\}$ may be considered the full group of characters of a certain commutative, compact, totally disconnected topological group G, just as the exponentials $\{e^{2\pi n x i}\}$ are the full character group of the reals modulo 1. The image of the group operation of G in the reals, $T_a(x) = x \dotplus a$, is introduced; $T_a(x)$, for fixed a, is a measure-preserving transformation of the unit interval into itself, so that we have an invariant integral $\int_0^1 f(x \dotplus a)dx = \int_0^1 f(x)dx$. After a discussion of the Lebesgue constants of the system $\{\Psi_n(x)\}$, the following uniqueness theorem is stated: If two Walsh series (not necessarily Fourier series) converge to the same function except perhaps on a countable set, then the series are identical.

2. *Convergence criteria for continued fractions*, by Professor V. F. Cowling, Lehigh University.

Recent years have seen a greatly renewed interest in the study of convergence criteria for continued fractions. The investigations of H. S. Wall, W. T. Scott and J. F. Paydon represent one approach. Another more geometric method was introduced by W. Leighton and W. J. Thron. This latter method has been considerably extended by Thron. For an account of this approach see the joint paper by Cowling, Leighton, and Thron (*Bull. Amer. Math. Soc.* vol. 50, 1944, pp. 351–357).

3. *Vector methods in the teaching of trigonometry and analytic geometry*, by Professor F. D. Murnaghan, Johns Hopkins University.

In this paper Professor Murnaghan advocates the use of vector methods in the teaching of trigonometry and analytic geometry. The idea of direction is basic in trigonometry, and the ideas of distance and direction are basic in geometry. These concepts are most conveniently specified by

the coördinates of a vector or, equivalently, by the direction numbers of a line segment. The central formula of trigonometry, namely, the formula for the cosine of the difference of two angles, follows immediately from the expression for the magnitude of the vector from one point on the unit circle to another. The idea of the positive and negative sides of a directed line or of an oriented plane is almost intuitive when the student is taught the significance of the coefficient vector of a linear equation in two or three variables. The concept of the alternating product of two plane vectors (or of three space vectors) leads naturally to the idea of the signed area of a parallelogram (or volume of a parallelepiped) and explains to the beginner the fundamental importance of determinants in analytic geometry. A treatment of analytic geometry from the point of view of vectors may be found in the speaker's book *Analytic Geometry* (Prentice-Hall, 1946).

4. *A few concepts in modern statistical inference*, by Professor S. S. Wilks, Princeton University.

The purpose here was to discuss some of the concepts of modern statistical inference without going into mathematical details. The concepts considered included random sampling, representative sampling, statistical control, acceptance sampling, confidence limits, tolerance limits, and statistical tests of significance. They were presented in conjunction with material on slides.

J. C. OXTOBY, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Summer Meeting, Madison, Wisconsin, September 6–7, 1948.

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
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KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 29, 1949
OHIO
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 27, 1948
ROCKY MOUNTAIN
SOUTHEASTERN, University of Alabama, University, March 18–19, 1949
SOUTHERN CALIFORNIA, John Muir Jr. College, Pasadena, March 12, 1949
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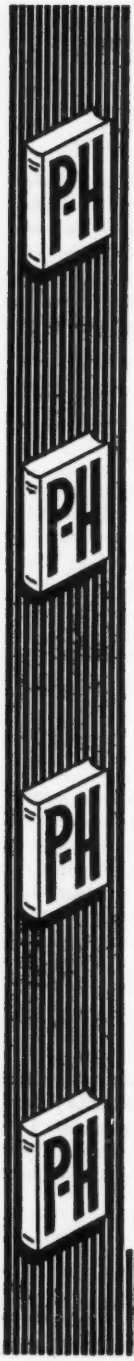

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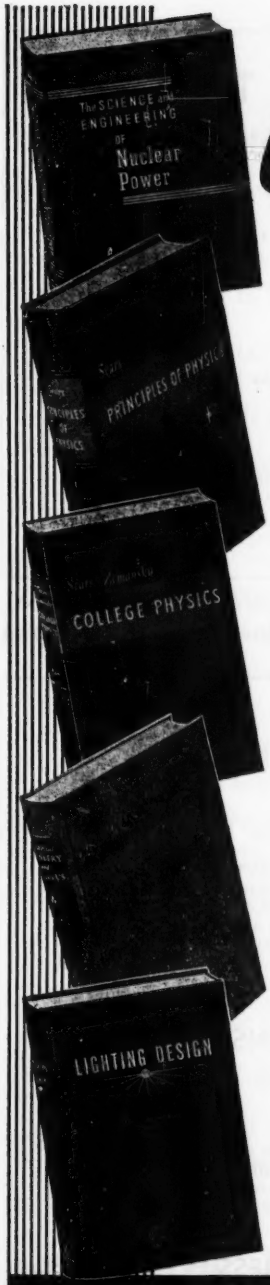

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